Dissipative forces in Lagrangian mechanics [mln9]

A dissipative force counteracts motion. Its direction is opposite to the direction of the velocity vector. Hence any dissipative force depends on velocity, be it on its direction only or also on its magnitude.

Dissipative forces are (by definition) non-conservative; they cannot be derived from a potential, not even from a velocity-dependent potential. However, Lagrangian mechanics allows the derivation of purely velocity dependent dissipative forces from a *dissipation function*.

Dissipative forces in Cartesian coordinates: $\mathbf{R}_i \doteq -h_i(v_i)\dfrac{\mathbf{v}_i}{v_i}, \quad i = 1, \ldots, N$

Transformation to generalized coordinates $q_1, \ldots, q_n$:

$$R_j = -\sum_{i=1}^{N} h_i(v_i)\frac{\mathbf{v}_i}{v_i}\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_{i=1}^{N} h_i(v_i)\frac{\mathbf{v}_i}{v_i}\cdot\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = -\sum_{i=1}^{N} h_i(v_i)\frac{\partial v_i}{\partial \dot{q}_j}.$$

We have used: $\dfrac{\partial \mathbf{r}_i}{\partial q_j} = \dfrac{\partial \mathbf{v}_i}{\partial \dot{q}_j}, \quad \mathbf{v}_i\cdot\dfrac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \dfrac{1}{2}\dfrac{\partial \mathbf{v}_i\cdot\mathbf{v}_i}{\partial \dot{q}_j} = \dfrac{1}{2}\dfrac{\partial v_i^2}{\partial \dot{q}_j} = v_i\,\dfrac{\partial v_i}{\partial \dot{q}_j}.$

Dissipation function: $P \doteq \displaystyle\sum_{i=1}^{N}\int_0^{v_i} dv_i\, h_i(v_i).$

$$\Rightarrow \; R_j = -\sum_{i=1}^{N} h_i(v_i)\frac{\partial v_i}{\partial \dot{q}_j} = -\frac{\partial}{\partial \dot{q}_j}\sum_{i=1}^{N}\int_0^{v_i} dv_i\, h_i(v_i) = -\frac{\partial P}{\partial \dot{q}_j}, \quad j = 1, \ldots, n$$

Lagrange equations: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} - \dfrac{\partial L}{\partial q_j} + \dfrac{\partial P}{\partial \dot{q}_j} = 0, \quad j = 1, \ldots, n.$

Common dissipative forces:

- kinetic friction (Coulomb damping): $\mathbf{R} = -\mu N\,\dfrac{\mathbf{v}}{v}.$
- linear damping (more common at low velocity): $\mathbf{R} = -\beta v\,\dfrac{\mathbf{v}}{v}.$
- quadratic damping (more common at high velocity): $\mathbf{R} = -\gamma v^2\,\dfrac{\mathbf{v}}{v}.$

Examples:

- Motion with friction on inclined plane [mex151]
- Linearly damped spherical pendulum [mex158]