Canonicity and Volume Preservation [mln90]

Illustration for one degree of freedom (2D phase space).

Consider transformation $(q,p) \to (Q,P)$.

Area preservation: Jacobian determinant $D = 1$ or, equivalently, area inside any closed path $\mathcal{C}$ is invariant.

Canonicity: There exists a generating function, e.g. $F_1(q,Q)$.

Canonicity implies area preservation:

- Given canonicity specified by $F_1(q,Q)$.
- $\Rightarrow\ p = \dfrac{\partial F_1}{\partial q}, \quad P = -\dfrac{\partial F_1}{\partial Q}.$
- $\Rightarrow\ \dfrac{\partial Q}{\partial q} = 0$ (Q and q are independent); $\dfrac{\partial P}{\partial p}$ is finite, in general.
- $\Rightarrow\ \dfrac{\partial P}{\partial q} = -\dfrac{\partial^2 F_1}{\partial Q \partial q} = -\left(\dfrac{\partial Q}{\partial p}\right)^{-1}.$
- $\Rightarrow\ D \doteq \dfrac{\partial(Q,P)}{\partial(q,p)} = \begin{vmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{vmatrix} = \dfrac{\partial Q}{\partial q}\dfrac{\partial P}{\partial p} - \dfrac{\partial Q}{\partial p}\dfrac{\partial P}{\partial q} = 1.$

Area preservation implies canonicity:

- Transformation: $p = p(q,Q) \quad P = P(q,Q)$.
- Area inside closed path $\mathcal{C}$: $\displaystyle\oint_{\mathcal{C}} p\,dq = \oint_{\mathcal{C}} P\,dQ.$
- $\Rightarrow\ \displaystyle\oint_{\mathcal{C}} [p(q,Q)dq - P(q,Q)dQ] = 0$ for arbitrary closed paths $\mathcal{C}$.
- $\Rightarrow$ Integrand must be perfect differential $dF_1(q,Q)$:

$$p\,dq - P\,dQ = dF_1 = \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial Q}dQ.$$

- $\Rightarrow\ p(q,Q) = \dfrac{\partial F_1}{\partial q}, \quad P(q,Q) = -\dfrac{\partial F_1}{\partial Q}.$