Infinitesimal canonical transformations [mln91]

Consider a canonical transformation generated by

$$F_2(q, P; \epsilon) = qP + \epsilon W(q, P; \epsilon),$$

where ϵ is a continuous parameter. The first term represents the *identity transformation*.

Transformed canonical coordinates:

$$Q(\epsilon) = q + \epsilon \frac{\partial W}{\partial P}, \qquad p(\epsilon) = P + \epsilon \frac{\partial W}{\partial q}.$$

Generator: $G(Q, P) \doteq \lim_{\epsilon \to 0} W(q, P; \epsilon)$ (Lie generating function).

Transformed canonical coordinates [to O(ϵ)]:

$$Q(\epsilon) = q + \epsilon \frac{\partial G}{\partial P}, \qquad P(\epsilon) = p - \epsilon \frac{\partial G}{\partial Q}.$$

Dependence of coordinates Q, P on ϵ expressed via two first-order ODEs:

$$\Rightarrow \ \frac{dQ}{d\epsilon} = \frac{\partial G}{\partial P}, \qquad \frac{dP}{d\epsilon} = -\frac{\partial G}{\partial Q}. \tag{1}$$

Solutions $Q(\epsilon)$, $P(\epsilon)$ with initial conditions $Q(0) = q$, $P(0) = p$.

For time evolution set $\epsilon = t$ and $G(Q, P) = H(Q, P)$.

The generator of the time evolution is the Hamiltonian and the differential equations (1) that determine this particular canonical transformation become the canonical equations:

$$\dot{Q} = \frac{\partial H}{\partial P}, \qquad \dot{P} = -\frac{\partial H}{\partial Q}.$$

The volume preservation of the time evolution in phase space is expressed by the Liouville theorem [tln45] [tln46].