Action-Angle Coordinates [mln92]

An elegant way of using Hamiltonian mechanics to solve a dynamical problem is to search for a canonical transformation to *action-angle* coordinates,

$$(q_1, \ldots, q_n; p_1, \ldots, p_n) \to (\underbrace{\theta_1, \ldots, \theta_n}_{\text{angles}}; \underbrace{J_1, \ldots, J_n}_{\text{actions}}),$$

such that the Hamiltonian turns into a function of the actions alone:

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n) \to K(J_1, \ldots, J_n).$$

If such a transformation exists and can be found then the solution of the canonical equations is simple:

$$\dot{J}_i = -\frac{\partial K}{\partial \theta_i} = 0 \quad \Rightarrow \; J_i = \text{const.}$$

$$\dot{\theta}_i = \frac{\partial K}{\partial J_i} \doteq \omega_i(J_1, \ldots, J_n) = \text{const.} \quad \Rightarrow \; \theta_i(t) = \omega_i t + \theta_i^{(0)}.$$

The inverse canonical transformation then yields $q_i(t)$ and $p_i(t)$.

Two or more degrees of freedom:
The existence of a transformation to action-angle coordinates is exceptional. Such systems are named *integrable*. Nonintegrable systems exhibit symptoms of *Hamiltonian chaos* (to be discussed later).

One degree of freedom:
Integrability is guaranteed. There exists a general prescription for finding the canonical transformation to action-angle coordinates.

The prescription for two modes of bounded motion is discussed in detail:

- libration (oscillation) [mln93],
- rotation [mln94].

The two modes are realized, for example, in the plane pendulum. The rotational motion can also be interpreted as unbounded motion in a periodic potential.