Actions and Angles for Rotations [mln94]

Hamiltonian: $H(q,p) = \dfrac{p^2}{2m} + V(q) = E = \text{const.}$ with $V(q+Q_0) = V(q)$.

Canonical momentum: $p(q,E) = \sqrt{2m\,[E - V(q)]}$.

Action J and Hamiltonian $K(J)$ from area A under trajectory:

$$A = \int_0^{Q_0} dq\, p(q,E) = \int_0^{Q_0} dq \sqrt{2m\,[E - V(q)]} = \int_0^{2\pi} d\theta\, J = 2\pi J.$$

$$\Rightarrow\ J(E) = \frac{1}{2\pi} \int_0^{Q_0} dq \sqrt{2m\,[E - V(q)]} \quad \Rightarrow\ E = K(J) = H(p,q).$$

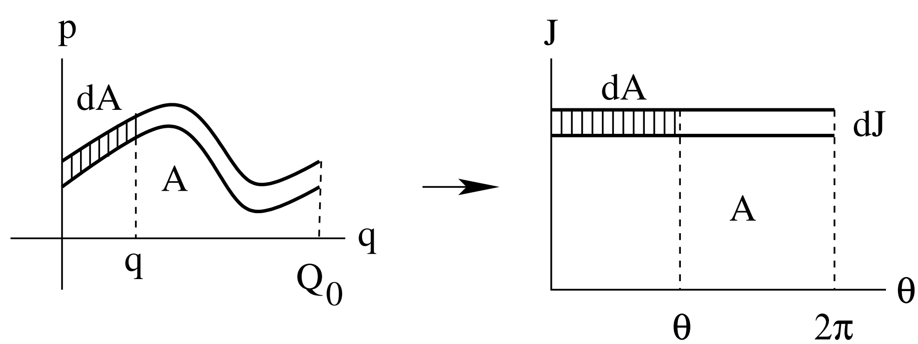
Angle variable $\theta(q,J)$ from area dA between nearby trajectories:

$$dA = \int_0^q dq\,[p(q,J+dJ) - p(q,J)] = dJ \int_0^q dq\, \frac{\partial}{\partial J} p(q,J) = dJ\theta(q,J).$$

$$\Rightarrow\ \theta(q,J) = \frac{\partial}{\partial J} \int_0^q dq\, p(q,J) = \frac{\partial}{\partial J} \int_0^q dq\, \sqrt{2m\,[K(J) - V(q)]}.$$

Time evolution: $J = \text{const.}$, $\theta(t) = \omega(J)t + \theta_0$, $\omega(J) = \dfrac{dK}{dJ}$.

$$\Rightarrow\ q(\theta,J) = q(t) \quad \Rightarrow\ p(q,J) = p(t).$$



In the case of rotations there is no natural boundary for J. Here J is only determined up to a constant.