Hamilton's Characteristic Function [mln97]

Two distinct ways of solving the Hamilton-Jacobi equation become available when the Hamiltonian does not explicitly depend on time.

If $H(q,p) = E = \text{const.}$ then $\dfrac{\partial S}{\partial t} = -E = \text{const.}$

Set $S(q,P,t) = W(q,P) - Et$.

Hamilton's characteristic function: $W(q_1,\ldots,q_n;P_1,\ldots,P_n)$.

Method #1:

- Solve the Hamilton-Jacobi equation $H\left(q, \dfrac{\partial S}{\partial q}\right) + \dfrac{\partial S}{\partial t} = 0.$
- Proceed as in [mln96] but use $S(q,P,t) = W(q,P) - Et$.
- One of the integration constants is reserved: $P_1 = E$.

Method #2:

- Solve the Hamilton-Jacobi equation $H\left(q, \dfrac{\partial W}{\partial q}\right) - E = 0.$
- $W(q,P)$ is a F_2-type generating function of a canonical transformation to action-angle coordinates with $P_1 = K(P) = E$.
- Canonical Equations: $\dot{Q}_j = \dfrac{\partial K}{\partial P_j} = \delta_{j1}, \quad \dot{P}_j = -\dfrac{\partial K}{\partial Q_j} = 0.$
- Solution: $P_j = \text{const.}, \quad Q_j = t\delta_{j1} + Q_j^{(0)}$.
- Transformation to original canonical coordinates:

$$Q_j = \frac{\partial}{\partial P_j} W(q,P), \quad p_j = \frac{\partial}{\partial q_j} W(q,P).$$

$$\Rightarrow\; q_j = q_j\left(Q^{(0)}, P, t\right), \quad p_j = p_j\left(Q^{(0)}, P, t\right).$$