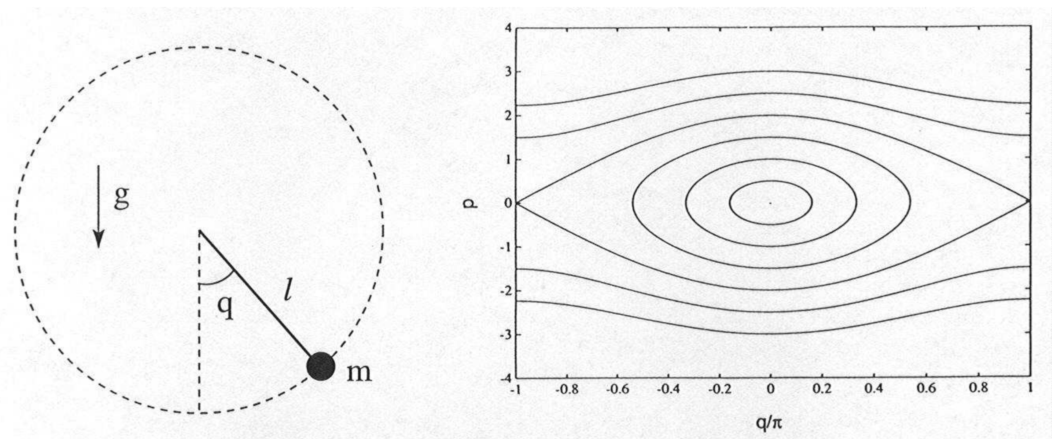
Integrabiliy as a Universal Property [mln98]

In autonomous Hamiltonian systems with one degree of freedom, integrability is always guaranteed. The evidence has previously been demonstrated in the contexts of Lagrangian and Hamiltonian mechanics.

Prototypical example: plane pendulum.

Hamiltonian: $H(q,p) = \dfrac{p^2}{2m} + mgl(1-\cos q)$.



Two-dimensional (2D) phase space.

$E =$ const. on sets of lines (1D).

Trajectories confined to $E =$ const.

Canonical transformation to action-angle coordinates:

$$
\begin{aligned}
(q,p) \quad &\Leftrightarrow \quad (\vartheta, J) \\
H(q,p) \quad &\Leftrightarrow \quad K(J) \\
\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \quad &\Leftrightarrow \quad \dot{\vartheta} = \frac{\partial K}{\partial J} \equiv \omega(J),\ \dot{J} = 0 \\
q(t),\ p(t) \quad &\Leftrightarrow \quad \vartheta(t) = \omega(J)t + \vartheta_0,\ J = \text{const.}
\end{aligned}
$$