Integrability as a Universal Property

In autonomous Hamiltonian systems with one degree of freedom, integrability is always guaranteed. The evidence has previously been demonstrated in the contexts of Lagrangian and Hamiltonian mechanics.

Prototypical example: plane pendulum.

Hamiltonian: \( H(q, p) = \frac{p^2}{2m} + mgl(1 - \cos q) \).

Two-dimensional (2D) phase space.

\( E = \text{const.} \) on sets of lines (1D).

Trajectories confined to \( E = \text{const.} \).

Canonical transformation to action-angle coordinates:

\[
(q, p) \quad \Leftrightarrow \quad (\vartheta, J) \\
H(q, p) \quad \Leftrightarrow \quad K(J) \\
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad \Leftrightarrow \quad \dot{\vartheta} = \frac{\partial K}{\partial J} \equiv \omega(J), \quad \dot{J} = 0 \\
q(t), \ p(t) \quad \Leftrightarrow \quad \vartheta(t) = \omega(J)t + \vartheta_0, \ J = \text{const.}
\]