Dynamical System with 2 Degrees of Freedom [msl15]

Newton's equation of motion: $m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$, $\quad \mathbf{x} = (x_1, x_2)$, $\quad \mathbf{y} \equiv (\dot{x}_1, \dot{x}_2)$

Velocity vector field in 4D phase space: $(\dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2)$.

Solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ describes trajectory in 4D phase space.
All trajectories are tangential to velocity vector field and nonintersecting.

Orbits are projections of trajectories onto (x_1, x_2)-plane.

Conservative force $\mathbf{F}(\mathbf{x})$: $\oint d\mathbf{s} \cdot \mathbf{F} = 0$ for all closed paths in (x_1, x_2)-plane.

Potential energy: $V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{s} \cdot \mathbf{F}$.

In conservative system, first integral of the motion guaranteed to exist:
$E(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + V(x_1, x_2)$.

Existence of second integral of the motion $K(x_1, x_2, \dot{x}_1, \dot{x}_2) = \text{const}$ guarantees *integrability* of dynamical system.

In *nonintegrable* systems, any trajectory in 4D phase space is confined to a 3D hypersurface $E = \text{const}$. In *integrable* systems, any trajectory is confined to the intersection of $E = \text{const}$ and $K = \text{const}$ in 4D phase space. The resulting 2D manifold has the topology of a *torus*.

Poincaré surface of section: Plot only those points of a trajectory where it crosses a particular hyperplane (e.g. $x_1 = 0$) in a particular direction (e.g. with $\dot{x}_1 > 0$).

On the Poincaré cut, lines represent quasiperiodic trajectories (on invariant tori) and fixed points represent periodic trajectories.

In *integrable* systems, the Poincaré maps of all trajectories are confined to lines. In *nonintegrable* systems, the Poincaré maps of quasiperiodic trajectories are confined to lines whereas the Poincaré maps of chaotic trajectories spread into 2D regions.

Nearby quasiperiodic trajectories move apart *linearly* in time.
Nearby chaotic trajectories move apart *exponentially* in time.