Orbits of Power-Law Potentials [msl21]

$$E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \tilde{V}(r), \quad \tilde{V}(r) = V(r) + \frac{\ell^2}{2mr^2}, \quad E > \tilde{V}(r) > V(r).$$

$$E - V(r) = \frac{1}{2}mv^2, \quad E - \tilde{V}(r) = \frac{1}{2}m\dot{r}^2, \quad \tilde{V}(r) - V(r) = \frac{1}{2}mr^2\dot{\vartheta}^2.$$

particle speed: $v \propto \sqrt{E - V}$.

radial speed: $|\dot{r}| \propto \sqrt{E - \tilde{V}}$.

angular speed: $r|\dot{\vartheta}| \propto \sqrt{\tilde{V} - V}$.

(i) $V(r) = -\dfrac{\kappa}{r^\alpha}, \quad 0 < \alpha < 2$:

$\tilde{V}(r)$ has minimum at $r_0 = (\alpha\kappa m/\ell^2)^{1/(\alpha-2)}$.

$E = E_1$: unbounded orbit, turning point ($\dot{r} = 0$) at $\tilde{V}(r_{min}) = E_1$.

$E = E_3$: bounded orbit, turning points at $\tilde{V}(r_{min}) = \tilde{V}(r_{max}) = E_3$.

$E = E_4$: circular orbit at r_0: $\dot{r} = 0$, $\dot{\vartheta} = \text{const.}$

(ii) $V(r) = -\dfrac{\kappa}{r^\alpha}, \quad \alpha > 2$:

$\tilde{V}(r)$ has maximum at $r_0 = (\alpha\kappa m/\ell^2)^{1/(\alpha-2)}$.

$E < \tilde{V}(r_0)$ and large r: unbounded orbit at $r > r_2$, where $\tilde{V}(r_2) = E$.

$E < \tilde{V}(r_0)$ and small r: bounded orbit at $r < r_1$, where $\tilde{V}(r_1) = E$.

$E > \tilde{V}(r_0)$: Unbounded orbit with particle spiraling through center.

$E = \tilde{V}(r_0)$: Unstable circular orbit exists.

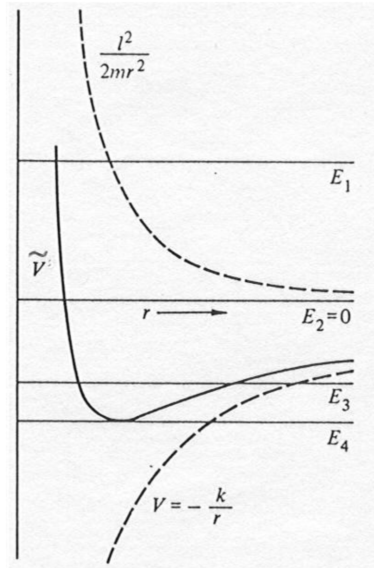
(iii) $V(r) = \kappa' r^{\alpha'}, \quad \kappa' = -\kappa > 0, \quad \alpha' = -\alpha > 0$:

$\tilde{V}(r)$ has minimum at $r_0 = (\ell^2/\alpha'\kappa' m)^{1/(\alpha'+2)}$.

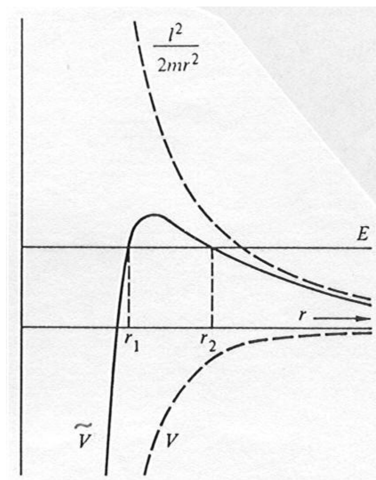
All orbits are bounded: $r_1 < r < r_2$, where $\tilde{V}(r_1) = \tilde{V}(r_2) = E$

$E = \tilde{V}(r_0)$: circular orbit exists.

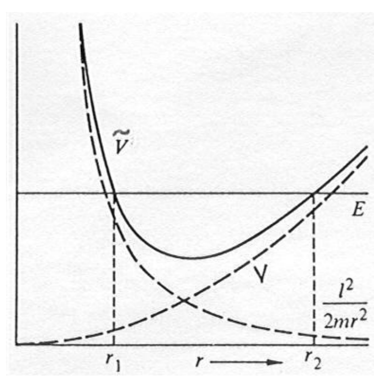
(i) $\alpha = 1$ (gravitation):



(ii) $\alpha = 3$:



(iii) $\alpha' = 2$ (harmonic oscillator):



[Goldstein 1981]