Scattering angle in the laboratory frame

The scattering experiment is performed in the laboratory frame.
- observed scattering angle: $\tilde{\theta}$,
- observed scattering cross section: $\tilde{\sigma}(\tilde{\theta})$,
- projectile of mass $m_1$ and target of mass $m_2$.

The theoretical analysis is performed in the center-of-mass frame:
- problem reduced to one degree of freedom,
- total mass $M = m_1 + m_2$,
- reduced mass $m = m_1 m_2 / (m_1 + m_2)$,
- calculated scattering angle: $\theta$,
- calculated scattering cross section: $\sigma(\theta)$.

Task #1: establish the relation between $\theta$ and $\tilde{\theta}$.

$$m_1 \tilde{v}_0 = (m_1 + m_2) \tilde{v}_{\text{cm}} \quad \Rightarrow \quad \tilde{v}_{\text{cm}} = \frac{m_1}{m_1 + m_2} \tilde{v}_0 = \frac{m}{m_2} \tilde{v}_0.$$

$$\bar{v}_1 \sin \bar{\theta} = v_1 \sin \theta, \quad \bar{v}_1 \cos \bar{\theta} = v_1 \cos \theta + \tilde{v}_{\text{cm}}.$$
Relative velocity after collision: \( \mathbf{v} = \mathbf{\tilde{v}}_2 - \mathbf{\tilde{v}}_1 = \mathbf{v}_2 - \mathbf{v}_1 \) (frame-independent).

Linear momentum in center-of-mass frame: \( m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = 0. \)

\[
\Rightarrow \mathbf{v}_1 = -\frac{m_2}{m_1 + m_2} \mathbf{v}, \quad \mathbf{v}_2 = \frac{m_1}{m_1 + m_2} \mathbf{v} \quad \Rightarrow m_1 \mathbf{v}_1 = m \mathbf{v}.
\]

\[
\Rightarrow \tan \bar{\theta} = \frac{v_1 \sin \theta}{v_1 \cos \theta + \tilde{v}_{cm}} = \frac{\sin \theta}{\cos \theta + \rho}, \quad \rho = \frac{m}{m_2} \frac{\tilde{v}_0}{v_1} = \frac{m_1 \tilde{v}_0}{m_2 v}.
\]

\[
\Rightarrow \cos \theta = -\rho(1 - \cos^2 \bar{\theta}) + \cos \bar{\theta} \sqrt{1 - \rho^2(1 - \cos^2 \bar{\theta})}.
\]

Elastic scattering: \( T = \frac{1}{2} m_1 \mathbf{\tilde{v}}_0^2 = \frac{1}{2} m v^2 \) (in center-of-mass frame)

\[
\Rightarrow \mathbf{\tilde{v}}_0 = v \quad \Rightarrow \rho = m_1/m_2.
\]

**Task #2**: establish the relation between \( \sigma \) and \( \bar{\sigma} \).

Number of particles scattered into infinitesimal solid angle:

\[
2\pi I \sigma(\theta) \sin \theta |d\theta| = 2\pi I \bar{\sigma}(\bar{\theta}) \sin \bar{\theta} |d\bar{\theta}|.
\]

\[
\Rightarrow \bar{\sigma}(\bar{\theta}) = \sigma(\theta) \frac{\sin \theta}{\sin \bar{\theta}} \left| \frac{d\theta}{d\bar{\theta}} \right| = \sigma(\theta) \left| \frac{d \cos \theta}{d \cos \bar{\theta}} \right|.
\]

\[
\Rightarrow \bar{\sigma}(\bar{\theta}) = \sigma(\theta) \left[ 2\rho \cos \bar{\theta} + \frac{1 + \rho^2 \cos(2\bar{\theta})}{\sqrt{1 - \rho^2 \sin^2 \bar{\theta}}} \right].
\]

Special case: elastic scattering between particles of equal mass:

\[
m_1 = m_2 \quad \Rightarrow \cos \theta = \cos(2\bar{\theta}) \quad \Rightarrow \bar{\theta} = \frac{\theta}{2}, \quad \bar{\sigma}(\bar{\theta}) = 4 \cos \frac{\theta}{2} \sigma(\theta).
\]