Poisson Brackets [msl30]

Definition: $\{f,g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j} \frac{\partial f}{\partial p_j} \right),$

where $f(q_1, \ldots, q_n, p_1, \ldots, p_n)$, $g(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are arbitrary dynamical variable expressed as functions of canonical coordinates.

Algebraic properties:

- $\{f,g\} = -\{g,f\}$,
- $\{f,c\} = 0 \quad \text{if} \quad c = \text{const.}$,
- $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$,
- $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$,
- $\frac{\partial}{\partial t}\{f,g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\}$,
- $\{q_j, f\} = \frac{\partial f}{\partial p_j}, \quad \{p_j, f\} = -\frac{\partial f}{\partial q_j}$.

Fundamental Poisson brackets: $\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$

Invariance under canonical transformations:

$$Q_j = Q_j(q_1, \ldots, q_n, p_1, \ldots, p_n), \;\; P_j = P_j(q_1, \ldots, q_n, p_1, \ldots, p_n)$$

$$\Rightarrow \;\; \{Q_i, Q_j\}_{q,p} = 0, \quad \{P_i, P_j\}_{q,p} = 0, \quad \{Q_i, P_j\}_{q,p} = \delta_{ij}$$

Canonical equations: $\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}.$

Jacobi's identity: $\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$

Poisson's theorem: $\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}$ [mex191]

Implication: If f and g are integrals of the motion, then $\{f,g\}$ is also an integral of the motion.