Poisson Brackets

Definition: \( \{ f, g \} = \sum_{j=1}^{n} \left( \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j} \frac{\partial f}{\partial p_j} \right) \),

where \( f(q_1, \ldots, q_n, p_1, \ldots, p_n) \), \( g(q_1, \ldots, q_n, p_1, \ldots, p_n) \) are arbitrary dynamical variables expressed as functions of canonical coordinates.

Algebraic properties:

- \( \{ f, g \} = -\{ g, f \} \)
- \( \{ f, c \} = 0 \) if \( c = \text{const.} \)
- \( \{ f_1 + f_2, g \} = \{ f_1, g \} + \{ f_2, g \} \)
- \( \{ f_1 f_2, g \} = f_1 \{ f_2, g \} + f_2 \{ f_1, g \} \)
- \( \frac{\partial}{\partial t} \{ f, g \} = \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\} \)
- \( \{ q_j, f \} = \frac{\partial f}{\partial p_j}, \quad \{ p_j, f \} = -\frac{\partial f}{\partial q_j} \)

Fundamental Poisson brackets: \( \{ q_i, q_j \} = 0, \quad \{ p_i, p_j \} = 0, \quad \{ q_i, p_j \} = \delta_{ij} \)

Invariance under canonical transformations:

\( Q_j = Q_j(q_1, \ldots, q_n, p_1, \ldots, p_n), \quad P_j = P_j(q_1, \ldots, q_n, p_1, \ldots, p_n) \)

\( \Rightarrow \{ Q_i, Q_j \}_{q,p} = 0, \quad \{ P_i, P_j \}_{q,p} = 0, \quad \{ Q_i, P_j \}_{q,p} = \delta_{ij} \)

Canonical equations: \( \dot{q}_i = \frac{\partial H}{\partial p_i} = \{ q_i, H \}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{ p_i, H \} \)

Jacobi's identity: \( \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \)

Poisson's theorem: \( \frac{d}{dt} \{ f, g \} = \left\{ \frac{df}{dt}, g \right\} + \left\{ f, \frac{dg}{dt} \right\} \quad [\text{mex191}] \)

Implication: If \( f \) and \( g \) are integrals of the motion, then \( \{ f, g \} \) is also an integral of the motion.