Robustness of apsidal angles

(a) Given the result of [mex126], namely that nearly circular orbits at radius \( r = R \) of a central force potential \( V(r) \) have apsidal angle \( \Delta \vartheta = \pi \sqrt{V'(R)/3V'(R) + RV''(R)} \), show that the only cases for which this apsidal angle is independent of the radius are the power-law potentials \( V(r) = -\kappa/r^\alpha, \alpha < 2, \alpha \neq 0 \) and the logarithmic potential \( V(r) = \kappa \ln r \). (b) Show that the value of the apsidal angle is \( \Delta \vartheta = \pi/\sqrt{2 - \alpha} \), where the value \( \alpha = 0 \) pertains to the logarithmic potential.

Solution: