[mex160] Bead sliding down cylindrical spiral

A bead of mass m slides down (from rest and without friction) a spiral with vertical axis: $z = a\phi,\ r = R$ in cylindrical coordinates.
(a) Write the Lagrangian $L(z, r, \phi, \dot{z}, \dot{r}, \dot{\phi})$ and the two equations of holonomic constraint, $f_j(z, r, \phi) = 0, j = 1, 2$. Derive the Lagrange equations.
(b) From the three Lagrange equations and the two equations of constraint determine the three coordinates $z(t), r(t), \phi(t)$ and the two Lagrange multipliers $\lambda_j(t), j = 1, 2$.
(c) Infer the generalized force of constraint for each cylindrical coordinate.
(d) Show that the results are consistent with $\dot{J}_z = N_z$, where J_z is the angular momentum of the bead and N_z is the torque exerted by the spiral on the bead.

Solution: