[mex165] Precession of the perihelion: orbital integral

Consider the Kepler problem with a correction term reflecting relativistic effects:

$$V(r) = -\frac{\kappa}{r} - \frac{\gamma}{r^3}, \qquad \kappa = GmM, \quad \gamma = \frac{G\ell^2 M}{c^2 m}.$$

Calculate the angle $\delta\vartheta$ of precession per cycle of the perihelion as a correction to the known apsidal angle of the Kepler problem in a perturbative treatment of the orbital integral:

$$\delta\vartheta = 2\int_{r_1}^{r_2} dr \, \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} - 2\pi.$$

Solution: