Precession of the perihelion: orbital differential equation

Consider the Kepler problem with a correction term reflecting relativistic effects:

\[ V(r) = -\frac{\kappa}{r} - \frac{\gamma}{r^3}, \quad \kappa = GmM, \quad \gamma = \frac{G\ell^2 M}{c^2 m}. \]

(a) Show that the resulting orbital differential equation acquires a nonlinear term as follows:

\[ \frac{d^2 u}{d\vartheta^2} + u = \frac{1}{p} + \alpha u^2, \quad \frac{1}{p} = \frac{m\kappa}{\ell^2}, \quad \alpha = \frac{3GM}{c^2}. \]

(b) Calculate the angle \( \delta \vartheta \) of precession per cycle of the perihelion by a perturbative solution.

Solution: