Quality factor of damped harmonic oscillator

(a) Consider the driven harmonic oscillator, \( m\ddot{x} = -kx - \gamma \dot{x} + F_0 \cos \omega t \), in a steady-state motion. Use the parameters \( \beta = \gamma / 2m \), \( \omega_0 = \sqrt{k/m} \), \( A = F_0 / m \). In [mex182] we have calculated the maximum (averaged) power input, \( P_{\text{max}} = \langle P(\omega_P) \rangle \), and in [mex181] we have calculated the average energy \( \langle E(\omega) \rangle \) stored in the oscillator. Determine the quality factor of the driven oscillator defined as \( Q = \frac{2\pi \langle E(\omega) \rangle}{\langle P(\omega) \rangle \tau} \) with \( \tau = \frac{2\pi}{\omega} \). Show that to leading order in \( \beta/\omega_0 \) the quality factor is equal to the amplitude ratio at resonance and at zero frequency: \( Q = D(\omega_R)/D(0) \).

(b) Consider the harmonic oscillator, \( m\ddot{x} = -kx - \gamma \dot{x} \), with weak damping (\( \beta/\omega_0 \ll 1 \)) and no driving force. Determine the quality factor \( Q \) of the damped oscillator defined as \( 2\pi \) times the ratio of the instantaneous energy stored, \( E(t) \), and the energy loss per period, \( \tau |dE/dt| \). Evaluate the result to leading order in \( \beta/\omega_0 \).

Solution: