[mex19] Hopf bifurcation

A simple Hopf bifurcation generates a limit cycle from a point attractor upon variation of some parameter in the equations of motion of a dynamical system. Consider the dynamical system specified (in polar coordinates) by

\[
\dot{r} = -\Gamma r - r^3, \quad \dot{\theta} = \omega,
\]

where \(\Gamma\) and \(\omega\) are constants.

(a) Find the exact solution \(r(t), \theta(t)\) for initial conditions \(r(0) = r_0, \theta(0) = 0\).

(b) Identify the circular periodic trajectory for \(\Gamma < 0\), which plays the role of a limit cycle, and determine its radius.

(c) Determine the nature of the fixed point at \(r = 0\) for \(\Gamma > 0\) and \(\Gamma < 0\).

(d) Plot three trajectories in the \((x, y)\)-plane to illustrate the emergence of a limit cycle from a stable fixed point. The first trajectory is for \(\Gamma > 0\). It will spiral into the point attractor at the origin. The second and third attractor are for \(\Gamma < 0\) with initial conditions inside and outside the limit cycle, respectively. Fine-tune the parameters and initial conditions to make the message of your graph compelling.

(e) Choose several values of \(t_{\text{max}}\) for fixed values of \(r_0, \omega, \Gamma\). Then plot \(r(t_{\text{max}})\) versus \(\Gamma\) to illustrate the emergence of a bifurcation singularity in the limit \(t_{\text{max}} \to \infty\). Again fine-tune your parameter values to optimize your graph for didactic effect.

Solution: