Driven harmonic oscillator with Coulomb damping

The harmonic oscillator with Coulomb damping and harmonic driving force is described by the equation of motion,

\[ \ddot{x} + \alpha \text{sgn}(\dot{x}) + \omega_0^2 x = A \cos(\omega t), \]  

where \( \omega_0^2 = k/m, \alpha = \mu/m, A = F_0/m. \) The function \( \text{sgn}(\dot{x}) \) denotes the sign (±) of the instantaneous velocity. The oscillator has mass \( m \) and the spring has stiffness \( k. \) The coefficient of kinetic (and static) friction is \( \mu. \) The natural angular frequency of oscillation is \( \omega_0. \) The harmonic driving force has amplitude \( F_0 \) and angular frequency \( \omega. \) In this project we consider an oscillator at resonance (\( \omega = \omega_0 = 1 \)) launched from \( x(0) = 0 \) with initial velocity \( \dot{x}(0) = v_0 > 0. \)

(a) Use the DSolve option of Mathematica to determine the analytic solutions of (1) with the given initial conditions, valid over a time interval with \( \dot{x} > 0. \) Check whether the solution that Mathematica gives you can be further simplified by hand. Use the ParametricPlot option of Mathematica to plot this solution in the phase plane, i.e. \( x \) versus \( \dot{x} \). Use \( A = 1, v_0 = 9 \) and various values of \( \alpha \) (all in SI units).

(b) Use the NDSolve and ParametricPlot options of Mathematica to generate and plot data for the solution of (1) over a larger time interval. Use again \( A = 1, v_0 = 9 \) and various values of \( \alpha. \) Tune \( \alpha \) to a value that yields a periodic trajectory.

(c) Investigate the stability of the periodic trajectory thus found numerically. Is it a limit cycle? Vary the initial conditions and check whether the periodic trajectory attracts or repels nearby trajectories.

Solution: