A bead of mass $m$ slides without friction along a wire of parabolic shape, $y = Ax^2$, in a uniform gravitational field $g$ pointing in the negative $y$-direction.

(a) Construct the Lagrangian $L(x, \dot{x})$.

(b) Derive the Lagrange equation for $x(t)$.

(c) Determine the angular frequency of small-amplitude oscillations.

(d) Construct the Hamiltonian $H(x, p_x)$ via Legendre transform and derive the canonical equations.

(e) Derive from the (alternative) Lagrangian $L(x, y, \dot{x}, \dot{y})$ and the equation of holonomic constraint, $f(x, y) = y - Ax^2 = 0$, three equations for the unknowns $x(t), y(t)$ (dynamical variables), and $\lambda(t)$ (Lagrange multiplier).

(f) Express the components $(Q_x, Q_y)$ of the normal force (of constraint) acting on the bead during its motion in terms of $x, \dot{x}, \ddot{x}$.

(g) Interpret the result for the normal force at $x = 0$ as a combined reaction force to gravity and centrifugation.

Solution: