[mex81] When does the Hamiltonian represent the total energy?

Consider a dynamical system with $3N$ degrees of freedom subject to $k$ holonomic constraints: $r_i = r_i(q_1, \ldots, q_n, t), i = 1, \ldots, N, n = 3N - k$. The kinetic and potential energies are given by the expressions

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{r}_i|^2, \quad V = V(r_1, \ldots, r_N, \dot{r}_1, \ldots, \dot{r}_N, t).$$

Show that the Hamiltonian $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ derived from these specifications is equal to the total energy, $E = T + V$, only if (i) the potential energy does not depend on the velocities $\dot{r}_i$ and (ii) if the holonomic constraints are not explicitly time-dependent.

Solution: