

[mex97] Hamilton-Jacobi equation for the harmonic oscillator

Determine the time evolution of the canonical coordinates $q(t), p(t)$ for the harmonic oscillator, $H(q, p) = p^2/2m + \frac{1}{2}m\omega_0^2 q^2$, by solving the Hamilton-Jacobi equation along two different avenues.

(a) Use the ansatz $S(q, E, t) = W(q, E) - Et$ for Hamilton's principal function. Solve the Hamilton-Jacobi equation for $S(q, E, t)$. Use $Q = \partial S/\partial E$ to derive $q(t)$ and $\partial S/\partial q$ to derive $p(t)$.

(b) Solve the Hamilton-Jacobi equation for Hamilton's principal function $W(q, E)$. Use $Q = \partial W/\partial E$ to derive $q(E, Q)$ and $\partial W/\partial q$ to derive $p(E, Q)$. Substitute these results into $H(q, p)$ to obtain the transformed Hamiltonian $K(E) = E$. Solve the canonical equations for the transformed canonical coordinates Q, E and substitute them into $q(E, Q)$ and $p(E, Q)$ to obtain $q(t), p(t)$.

Solution: