The first and second laws of thermodynamics imply that
\[ dU = TdS + YdX + \mu dN \]  
(1)
with
\[ \left( \frac{\partial U}{\partial S} \right)_{X,N} = T, \quad \left( \frac{\partial U}{\partial X} \right)_{S,N} = Y, \quad \left( \frac{\partial U}{\partial N} \right)_{S,X} = \mu \]
is the exact differential of a function \( U(S, X, N) \).

Here \( X \) stands for \( V, M, \ldots \) and \( Y \) stands for \( -p, H, \ldots \).

Note: for irreversible processes \( dU < TdS + YdX + \mu dN \) holds.

\( U, S, X, N \) are extensive state variables.

\( U(S, X, N) \) is a 1\textsuperscript{st} order homogeneous function: \( U(\lambda S, \lambda X, \lambda N) = \lambda U(S, X, N) \).

\[
U[(1 + \epsilon)S, (1 + \epsilon)X, (1 + \epsilon)N] = U + \frac{\partial U}{\partial S} \epsilon S + \frac{\partial U}{\partial X} \epsilon X + \frac{\partial U}{\partial N} \epsilon N = (1 + \epsilon)U.
\]

Euler equation:
\[ U = TS + YX + \mu N. \]  
(2)

Total differential of (2):
\[ dU = TdS + SdT + YdX + XdY + \mu dN + N d\mu \]  
(3)

Subtract (1) from (3):
Gibbs-Duhem equation: \( SdT + XdY + N d\mu = 0. \)

The Gibbs-Duhem equation expresses a relationship between the intensive variables \( T, Y, \mu \). It can be integrated, for example, into a function \( \mu(T, Y) \).

Note: a system specified by \( m \) independent extensive variables possesses \( m - 1 \) independent intensive variables.

Example for \( m = 3 \): \( S, V, N \) (extensive); \( S/N, V/N \) or \( p, T \) (intensive).

Complete specification of a thermodynamic system must involve at least one extensive variable.