

Useful relations between partial derivatives [tln6]

Consider state variables x, y, z, w . Only two of the variables are independent.

$$\#1 \quad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial y}{\partial x}\right)_z^{-1}$$

$$\#2 \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

$$\#3 \quad \left(\frac{\partial x}{\partial w}\right)_z = \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial w}\right)_z$$

$$\#4 \quad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial y}\right)_w + \left(\frac{\partial x}{\partial w}\right)_y \left(\frac{\partial w}{\partial y}\right)_z$$

Proof:

Start with total differential of $x(y, z)$: $dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$.

Substitute total differential of $y(x, z)$: $dy = \left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz$.

$$\Rightarrow \left[\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z - 1 \right] dx + \left[\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x + \left(\frac{\partial x}{\partial z}\right)_y \right] dz = 0.$$

Expressions in brackets must vanish independently \Rightarrow #1 and #2.

Introduce parameter w : $x(y, z)$ with $y = y(w)$ and $z = z(w)$.

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz \Rightarrow \frac{dx}{dw} = \left(\frac{\partial x}{\partial y}\right)_z \frac{dy}{dw} + \left(\frac{\partial x}{\partial z}\right)_y \frac{dz}{dw}.$$

Specify path: $z = \text{const}$ i.e. $dz = 0$: \Rightarrow #3.

Introduce parameter y : $x(y, w)$ with $w = w(y)$ and $z = z(y)$.

$$dx = \left(\frac{\partial x}{\partial y}\right)_w dy + \left(\frac{\partial x}{\partial w}\right)_y dw \Rightarrow \frac{dx}{dy} = \left(\frac{\partial x}{\partial y}\right)_w + \left(\frac{\partial x}{\partial w}\right)_y \frac{dw}{dy}.$$

Specify path: $z = \text{const}$ i.e. $dz = 0$: \Rightarrow #4.