Partition function of ideal quantum gases [tln63]

Canonical partition function: $Z_N = \sum_{\{n_k\}}' \sigma(n_1, n_2, \ldots) \exp\left(-\beta \sum_{k=1}^{\infty} n_k \epsilon_k\right)$.

$\sum_{\{n_k\}}'$: sum over all occupation numbers compatible with $\sum_{k=1}^{\infty} n_k = N$.

The statistical weight factor $\sigma(n_1, n_2, \ldots)$ is different for fermions and bosons:

- Bose-Einstein statistics: $\sigma_{BE}(n_1, n_2, \ldots) = 1$ for arbitrary values of n_k.
- Fermi-Dirac statistics: $\sigma_{FD}(n_1, n_2, \ldots) = \begin{cases} 1 & \text{if all } n_k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$.

What is the statistical weight factor for the Maxwell-Boltzmann gas?

$$Z_N = \frac{1}{N!}\tilde{Z}^N = \frac{1}{N!}\left(\sum_{k=1}^{\infty} e^{-\beta\epsilon_k}\right)^N = \frac{1}{N!}\sum_{\{n_k\}}' \frac{N!}{n_1! n_2! \ldots}\left(e^{-\beta\epsilon_1}\right)^{n_1}\left(e^{-\beta\epsilon_2}\right)^{n_2}\cdots$$
$$= \sum_{\{n_k\}}' \frac{1}{n_1! n_2! \ldots} \exp\left(-\beta \sum_{k=1}^{\infty} n_k \epsilon_k\right).$$

- Maxwell-Boltzmann statistics: $\sigma_{MB}(n_1, n_2, \ldots) = \frac{1}{n_1! n_2! \ldots}$.

Grandcanonical partition function:

$$\Rightarrow \; Z = \sum_{N=0}^{\infty} z^N Z_N = \sum_{\{n_k\}} \sigma(n_1, n_2, \ldots) \exp\left(-\beta \sum_{k=1}^{\infty} n_k(\epsilon_k - \mu)\right),$$

where we have used $z^N = (e^{\beta\mu})^N = \exp\left(\beta\mu \sum_{k=1}^{\infty} n_k\right)$.

- $Z_{BE} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots \exp\left(-\beta\sum_{k=1}^{\infty} n_k(\epsilon_k - \mu)\right) = \prod_{k=1}^{\infty}\left(1 - ze^{-\beta\epsilon_k}\right)^{-1}$.
- $Z_{FD} = \sum_{n_1=0}^{1}\sum_{n_2=0}^{1}\cdots \exp\left(-\beta\sum_{k=1}^{\infty} n_k(\epsilon_k - \mu)\right) = \prod_{k=1}^{\infty}\left(1 + ze^{-\beta\epsilon_k}\right)$.
- $Z_{MB} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots \frac{1}{n_1! n_2! \ldots} \exp\left(-\beta\sum_{k=1}^{\infty} n_k(\epsilon_k - \mu)\right) = \prod_{k=1}^{\infty}\exp\left(ze^{-\beta\epsilon_k}\right)$.