

## Ideal quantum gases: grand potential and thermal averages [tln64]

Grand potential:  $\Omega(T, V, \mu) = -k_B T \ln Z = U - TS - \mu \mathcal{N} = -pV$ .

- $\Omega_{MB} = -k_B T \sum_{k=1}^{\infty} z e^{-\beta \epsilon_k} = -k_B T \sum_{k=1}^{\infty} e^{-\beta(\epsilon_k - \mu)}$ ,
- $\Omega_{BE} = k_B T \sum_{k=1}^{\infty} \ln(1 - z e^{-\beta \epsilon_k}) = k_B T \sum_{k=1}^{\infty} \ln(1 - e^{-\beta(\epsilon_k - \mu)})$ ,
- $\Omega_{FD} = -k_B T \sum_{k=1}^{\infty} \ln(1 + z e^{-\beta \epsilon_k}) = -k_B T \sum_{k=1}^{\infty} \ln(1 + e^{-\beta(\epsilon_k - \mu)})$ .

Parametric representation [ $a = 1$  (FD),  $a = 0$  (MB),  $a = -1$  (BE)]:

$$\ln Z = \frac{pV}{k_B T} = \frac{1}{a} \sum_{k=1}^{\infty} \ln(1 + a z e^{-\beta \epsilon_k}).$$

Average number of particles:

$$\mathcal{N} = - \left( \frac{\partial \Omega}{\partial \mu} \right)_{T, V} = \frac{1}{\beta} \left( \frac{\partial \ln Z}{\partial \mu} \right)_{T, V} = \sum_{k=1}^{\infty} \frac{1}{z^{-1} e^{\beta \epsilon_k} + a} = \sum_{k=1}^{\infty} \langle n_k \rangle.$$

Average energy (internal energy):

$$U = - \left( \frac{\partial \ln Z}{\partial \beta} \right)_{z, V} = \sum_{k=1}^{\infty} \frac{\epsilon_k}{z^{-1} e^{\beta \epsilon_k} + a} = \sum_{k=1}^{\infty} \epsilon_k \langle n_k \rangle.$$

Average occupation number of energy level  $\epsilon_k$ :

$$\langle n_k \rangle = -\beta^{-1} \frac{\partial \ln Z}{\partial \epsilon_k} = \frac{1}{e^{\beta(\epsilon_k - \mu)} + a}.$$

Fluctuations in occupation number [tex110]:

$$\langle n_k^2 \rangle - \langle n_k \rangle^2 = \beta^{-2} \frac{\partial^2 \ln Z}{\partial \epsilon_k^2}.$$