Ideal Bose-Einstein gas: equation of state and internal energy [tln67]

Conversion of sums into integrals by means of density of energy levels [tex113]:

$$D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)} \left(\frac{m}{2\pi\hbar^2}\right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}, \quad V = L^{\mathcal{D}}.$$

Fundamental thermodynamic relations for BE gas:

$$\frac{pV}{k_BT} = -\sum_k \ln\left(1 - ze^{-\beta\epsilon_k}\right) = -\int_0^\infty d\epsilon\, D(\epsilon) \ln\left(1 - ze^{-\beta\epsilon}\right) = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z),$$

$$\mathcal{N} = \sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)}{z^{-1}e^{\beta\epsilon} - 1} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2}(z), \quad z < 1,$$

$$U = \sum_k \frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)\epsilon}{z^{-1}e^{\beta\epsilon} - 1} = \frac{\mathcal{D}}{2} k_BT \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z).$$

Warning: The range of fugacity is limited to the interval $0 \leq z \leq 1$. At $z = 1$, the expression for $\mathcal{N}$ must be amended by an additive term $z/(1-z)$ to account for the possibility of a macroscopic population of the lowest energy level (at $\epsilon = 0$). This amendment is only necessary for dimensionalities $\mathcal{D} > 2$, i.e. for the cases with $\lim_{\epsilon\to 0} D(\epsilon) = 0$.

Equation of state (with fugacity z in the role of parameter):

$$\frac{pV}{\mathcal{N}k_BT} = \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}, \quad z < 1.$$