Ideal Bose-Einstein gas: 
equation of state and internal energy

Conversion of sums into integrals by means of density of energy levels:

\[ D(\epsilon) = \frac{V}{\Gamma(D/2)} \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} e^{D/2-1}, \quad V = L^D. \]

Fundamental thermodynamic relations for BE gas:

\[ \frac{pV}{k_B T} = -\sum_k \ln (1 - z e^{-\beta \epsilon_k}) = -\int_0^\infty d\epsilon D(\epsilon) \ln (1 - z e^{-\beta \epsilon}) = \frac{V}{\lambda_T^D} g_{D/2+1}(z), \]

\[ N = \sum_k \frac{1}{z^{-1} e^{\beta \epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)}{z^{-1} e^{\beta \epsilon} - 1} = \frac{V}{\lambda_T^D} g_{D/2}(z), \quad z < 1, \]

\[ U = \sum_k \frac{\epsilon_k}{z^{-1} e^{\beta \epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon) \epsilon}{z^{-1} e^{\beta \epsilon} - 1} = \frac{D}{2} k_B T V \lambda_T^D g_{D/2+1}(z). \]

Warning: The range of fugacity is limited to the interval \( 0 \leq z \leq 1 \). At \( z = 1 \), the expression for \( N \) must be amended by an additive term \( z/(1-z) \) to account for the possibility of a macroscopic population of the lowest energy level (at \( \epsilon = 0 \)). This amendment is only necessary for dimensionalities \( D > 2 \), i.e. for the cases with \( \lim_{\epsilon\to 0} D(\epsilon) = 0 \).

Equation of state (with fugacity \( z \) in the role of parameter):

\[ \frac{pV}{N k_B T} = \frac{g_{D/2+1}(z)}{g_{D/2}(z)}, \quad z < 1. \]