Legendre transform [tln77]

Given is a function $f(x)$ with monotonic derivative $f'(x)$. The goal is to replace the independent variable x by $p = f'(x)$ with no loss of information.

Note: The function $G(p) = f(x)$ with $p = f'(x)$ is, in general, not invertible.

The Legendre transform solves this task elegantly.

- Forward direction: $g(p) = f(x) - xp$ with $p = f'(x)$.
- Reverse direction: $f(x) = g(p) + px$ with $x = -g'(p)$

Example 1: $f(x) = x^2 + 1$.

- $f(x) = x^2 + 1 \quad \Rightarrow \quad f'(x) = 2x \quad \Rightarrow \quad x = \dfrac{p}{2} \quad \Rightarrow \quad g(p) = 1 - \dfrac{p^2}{4}$.
- $g(p) = 1 - \dfrac{p^2}{4} \quad \Rightarrow \quad g'(p) = -\dfrac{p}{2} \quad \Rightarrow \quad p = 2x \quad \Rightarrow \quad f(x) = x^2 + 1$.

Example 2: $f(x) = e^{2x}$.

- $f(x) = e^{2x} \quad \Rightarrow \quad f'(x) = 2e^{2x} = p \quad \Rightarrow \quad x = \dfrac{1}{2}\ln\dfrac{p}{2}$
  $\Rightarrow \quad g(p) = \dfrac{p}{2} - \dfrac{p}{2}\ln\dfrac{p}{2}$.
- $g(p) = \dfrac{p}{2} - \dfrac{p}{2}\ln\dfrac{p}{2} \quad \Rightarrow \quad g'(p) = -\dfrac{1}{2}\ln\dfrac{p}{2} = -x$
  $\Rightarrow \quad p = 2e^{2x} \quad \Rightarrow \quad f(x) = e^{2x}$.