Bose–Einstein functions [tsl36]

$$g_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx\; x^{n-1}}{z^{-1}e^x - 1} = \sum_{l=1}^\infty \frac{z^l}{l^n}, \qquad 0 \leq z \leq 1.$$

Special cases:

$$g_0(z) = \frac{z}{1-z}, \quad g_1(z) = -\ln(1-z), \quad g_\infty(z) = z.$$

Riemann zeta function:

$$g_n(1) = \zeta(n) \doteq \sum_{l=1}^\infty \frac{1}{l^n}.$$

Special values:

$$\zeta(1) \to \infty, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Recurrence relation:

$$z g_n'(z) = g_{n-1}(z), \qquad n \geq 1.$$

Singularity at $z = 1$ for non-integer n:

$$g_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!}\zeta(n-\ell)\alpha^\ell, \qquad \alpha \doteq -\ln z.$$