Ideal Bose-Einstein gas: heat capacity [tsl41]

Internal energy:

$$\frac{U}{\mathcal{N}k_BT_v} = \begin{cases} \dfrac{\mathcal{D}}{2}\dfrac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}\dfrac{T}{T_v}, & T \geq T_c, \\ \dfrac{\mathcal{D}}{2}\zeta(\mathcal{D}/2+1)\left(\dfrac{T}{T_v}\right)^{\mathcal{D}/2+1}, & T \leq T_c. \end{cases}$$

Heat capacity at $T \geq T_c$ [use $zg'_n(z) = g_{n-1}(z)$ for $n \geq 1$]:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4}\right)\frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4}\frac{g'_{\mathcal{D}/2+1}(z)}{g'_{\mathcal{D}/2}(z)}.$$

Heat capacity at $T \leq T_c$:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2}+\frac{\mathcal{D}^2}{4}\right)\zeta\left(\frac{\mathcal{D}}{2}+1\right)\left(\frac{T}{T_v}\right)^{\mathcal{D}/2} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4}\right)\frac{\zeta\left(\frac{\mathcal{D}}{2}+1\right)}{\zeta\left(\frac{\mathcal{D}}{2}\right)}\left(\frac{T}{T_c}\right)^{\mathcal{D}/2}.$$

High-temperature asymptotic behavior:

$$\frac{C_V}{\mathcal{N}k_B} \sim \frac{\mathcal{D}}{2}\left[1 + \frac{\mathcal{D}/2-1}{2^{\mathcal{D}/2+1}}\left(\frac{T_v}{T}\right)^{\mathcal{D}/2}\right].$$