Fermi-Dirac functions [tsl42]

$$f_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx\ x^{n-1}}{z^{-1}e^x + 1}, \quad 0 \leq z < \infty$$

Series expansion:

$$f_n(z) = \sum_{l=1}^{\infty} (-1)^{l-1} \frac{z^l}{l^n}, \quad 0 \leq z \leq 1.$$

Special cases:

$$f_0(z) = \frac{z}{1+z}, \qquad f_1(z) = \ln(1+z), \qquad f_\infty(z) = z.$$

Recurrence relation:

$$z f_n'(z) = f_{n-1}(z), \quad n \geq 1.$$

Asymptotic expansion for $z \gg 1$:

$$\begin{aligned} f_n(z) &= \frac{(\ln z)^n}{\Gamma(n+1)} \left[1 + \sum_{k=2,4,\ldots} 2n(n-1)\cdots(n-k+1) \left(1 - \frac{1}{2^{k-1}}\right) \frac{\zeta(k)}{(\ln z)^k} \right] \\ &= \frac{(\ln z)^n}{\Gamma(n+1)} \left[1 + n(n-1)\frac{\pi^2}{6}(\ln z)^{-2} \right. \\ &\qquad \left. + n(n-1)(n-3)\frac{7\pi^4}{360}(\ln z)^{-4} + \ldots \right] \end{aligned}$$