[tex102] FD gas in $\mathcal{D}$ dimensions: ground-state energy

Given are the following expressions for the average number of particles, the average energy, the average occupation number at $T = 0$, and the density of states for an ideal Fermi-Dirac gas in $\mathcal{D}$ dimensions:

$$\mathcal{N} = \sum_k \langle n_k \rangle, \quad U = \sum_k \langle n_k \rangle \epsilon_k, \quad \langle n_k \rangle = \Theta(\epsilon_F - \epsilon_k), \quad D(\epsilon) = \frac{gV}{\Gamma(\mathcal{D}/2)} \left(\frac{2\pi m}{h^2} \right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}.$$

Derive from these expressions the following results for the dependence of the ground-state energy per particle, $U_0/\mathcal{N}$, on the Fermi energy ϵ_F and for the dependence of the ground-state energy density U_0/V on the particle density $\mathcal{N}/V$:

$$\frac{U_0}{\mathcal{N}} = \frac{\mathcal{D}}{\mathcal{D}+2} \epsilon_F, \qquad \frac{U_0}{V} \propto \left(\frac{\mathcal{N}}{V} \right)^{(\mathcal{D}+2)/\mathcal{D}}.$$

Solution: