[tex109] Fluctuations in a magnetic system

Consider a quantum magnet. The Hamiltonian is of the form $H = H_{int} - hm$, where H_{int} describes the (unspecified) interaction between microscopic magnetic moments, h is the magnitude of the external magnetic field (assumed constant) and m is the component of the total magnetic moment in the direction of the field. Given the Gibbs free energy $G(T, h, N) = -k_B T \ln Z_N$ as derived from the canonical partition function $Z_N = \mathrm{Tr}\, e^{-\beta H}$, where $\beta = (k_B T)^{-1}$, derive the following relations (a) between energy fluctuations and heat capacity at constant field,

$$\langle H^2 \rangle - \langle H \rangle^2 = \frac{\partial^2}{\partial \beta^2} \ln Z_N = k_B T^2 C_h,$$

and (b) between magnetisation fluctuations and isothermal susceptibility,

$$\langle m^2 \rangle - \langle m \rangle^2 = \beta^{-2} \frac{\partial^2}{\partial h^2} \ln Z_N = k_B T \chi_T.$$

Solution: