[tex116] BE gas in $D$ dimensions V: heat capacity at low temperature

The internal energy of the ideal Bose-Einstein gas in $D > 2$ dimensions and at $T \leq T_c$ is given by the following expression:

$$\frac{U}{Nk_BT_v} = \frac{D}{2}\zeta(D/2 + 1)\left(\frac{T}{T_v}\right)^{D/2+1}$$

(a) Use this result to derive the following expression for the heat capacity $C_V = (\partial U/\partial T)_V_N$:

$$\frac{C_V}{Nk_B} = \left(\frac{D}{2} + \frac{D^2}{4}\right)\frac{\zeta\left(\frac{D}{2} + 1\right)}{\zeta\left(\frac{D}{2}\right)}\left(\frac{T}{T_c}\right)^{D/2},$$

where $T_c = T_v[\zeta(D/2)]^{-2/D}$ is the critical temperature and $k_BT_v = \Lambda/v^{2/D}$ with $v = V/N$ and $\Lambda \equiv h^2/2\pi m$ a convenient reference temperature. (b) Show that the heat capacity is continuous at $T = T_c$ if $D \leq 4$ and discontinuous if $D > 4$. Find the discontinuity $\Delta C_V/Nk_B$ as a function of $D$ for $D > 4$. (c) Infer from the result of [tex97] the leading singularity of $C_V/Nk_B$ at $T/T_v \ll 1$ for $D = 1$ and $D = 2$. Then show that these singularities are consistent with the expression for $C_V/Nk_B$ obtained here in part (a) provided we substitute $(T_v/T_c)^{D/2} = \zeta(D/2)$.

Solution: