Quantum harmonic oscillators (microcanonical ensemble I)

Consider an array of $N$ quantum harmonic oscillators:

$$H = \sum_{i=1}^{N} \left[ \hbar \omega \left( n_i + \frac{1}{2} \right) \right], \quad n_i = 0, 1, 2, \ldots.$$ 

(a) Calculate the entropy $S(U, N)$ of this system in the microcanonical ensemble via combinatorics as follows: Set $U = U_0 + M\hbar\omega$, $U_0 = \frac{1}{2}N\hbar\omega$, $M = n_1 + \cdots + n_N$. Next determine the number $N_{\Delta}(M, N)$ of configurations $(n_1, \cdots, n_N)$ for fixed values of $M, N$. Then relate $S$ to $N_{\Delta}$.

(b) Derive the internal energy $U(T, N)$, and the heat capacity $C = (\partial U/\partial T)_N$ from $S(U, N)$.

Solution: