Contents of this Document [ntc4]

Probability Distribution [nln46]

Experiment represented by events in a sample space: $S = \{A_1, A_2, \ldots\}$.

Measurements represented by stochastic variable: $X = \{x_1, x_2, \ldots\}$.

Maximum amount of information experimentally obtainable is contained in the probability distribution:

$$P_X(x_i) \geq 0, \quad \sum_i P_X(x_i) = 1.$$

Partial information is contained in moments,

$$\langle X^n \rangle = \sum_i x_i^n P_X(x_i), \quad n = 1, 2, \ldots,$$

or cumulants (as defined in [nln47]),

- $\langle\langle X \rangle\rangle = \langle X \rangle$ (mean value)
- $\langle\langle X^2 \rangle\rangle = \langle X^2 \rangle - \langle X \rangle^2$ (variance)
- $\langle\langle X^3 \rangle\rangle = \langle X^3 \rangle - 3\langle X \rangle\langle X^2 \rangle + 2\langle X \rangle^3$

The variance is the square of the standard deviation: $\langle\langle X^2 \rangle\rangle = \sigma_X^2$.

For continuous stochastic variables we have

$$P_X(x) \geq 0, \quad \int dx P_X(x) = 1, \quad \langle X^n \rangle = \int dx\, x^n P(x).$$

In the literature $P_X(x)$ is often named 'probability density' and the term 'distribution' is used for

$$F_X(x) = \sum_{x_i < x} P_X(x_i) \quad \text{or} \quad F_X(x) = \int_{-\infty}^{x} dx' P_X(x')$$

in a cumulative sense.

Characteristic Function [nln47]

Fourier transform: $\Phi_X(k) \doteq \left\langle e^{ikx} \right\rangle = \int_{-\infty}^{+\infty} dx\, e^{ikx} P_X(x).$

Attributes: $\Phi_X(0) = 1, \quad |\Phi_X(k)| \leq 1.$

Moment generating function:

$$\Phi_X(k) = \int_{-\infty}^{+\infty} dx\, P_X(x) \left[\sum_{n=0}^{\infty} \frac{(ik)^n}{n!} x^n \right] = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle X^n \rangle$$

$$\Rightarrow \; \langle X^n \rangle \doteq \int_{-\infty}^{+\infty} dx\, x^n P_X(x) = (-i)^n \frac{d^n}{dk^n} \Phi_X(k) \bigg|_{k=0}.$$

Cumulant generating function:

$$\ln \Phi_X(k) \doteq \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle\langle X^n \rangle\rangle \quad \Rightarrow \; \langle\langle X^n \rangle\rangle = (-i)^n \frac{d^n}{dk^n} \ln \Phi_X(k) \bigg|_{k=0}.$$

Cumulants in terms of moments (with $\Delta X \doteq X - \langle X \rangle$): [nex126]

- $\langle\langle X \rangle\rangle = \langle X \rangle$
- $\langle\langle X^2 \rangle\rangle = \langle X^2 \rangle - \langle X \rangle^2 = \langle (\Delta X)^2 \rangle$
- $\langle\langle X^3 \rangle\rangle = \langle (\Delta X)^3 \rangle$
- $\langle\langle X^4 \rangle\rangle = \langle (\Delta X)^4 \rangle - 3\langle (\Delta X)^2 \rangle^2$

Theorem of Marcienkiewicz:
$\ln \Phi_X(k)$ can only be a polynomial if the degree is $n \leq 2$.

- $n = 1$: $\ln \Phi_X(k) = ika \quad \Rightarrow \; P_X(x) = \delta(x - a)$
- $n = 2$: $\ln \Phi_X(k) = ika - \frac{1}{2} bk^2 \quad \Rightarrow \; P_X(x) = \frac{1}{\sqrt{2\pi b}} \exp\left(-\frac{(x-a)^2}{2b} \right)$

Consequence: any probability distribution has either one, two, or infinitely many non-vanishing cumulants.

[nex126] Cumulants expressed in terms of moments

The characteristic function $\Phi_X(k)$ of a probability distribution $P_X(x)$, obtained via Fourier transform as described in [nln47], can be used to generate the moments $\langle X^n \rangle$ and the cumulants $\langle\langle X^n \rangle\rangle$ via the expansions

$$\Phi_X(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle X^n \rangle, \qquad \ln \Phi_X(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle\langle X^n \rangle\rangle.$$

Use these relations to express the first four cumulants in terms of the first four moments. The results are stated in [nln47]. Describe your work in some detail.

Solution:

Generating function [nln48]

The generating function $G_X(z)$ is a representation of the characteristic function $\Phi_X(k)$ that is most commonly used, along with factorial moments and factorial cumulants, if the stochastic variable X is integer valued.

Definition: $G_X(z) \doteq \langle z^x \rangle$ with $|z| = 1$.

Application to continuous and discrete (integer-valued) stochastic variables:

$$G_X(z) = \int dx\, z^x P_X(x), \qquad G_X(z) = \sum_n z^n P_X(n).$$

Definition of factorial moments:

$$\langle X^m \rangle_f \doteq \langle X(X-1)\cdots(X-m+1)\rangle, \quad m \geq 1; \quad \langle X^0 \rangle_f \doteq 0.$$

Function generating factorial moments:

$$G_X(z) = \sum_{m=0}^{\infty} \frac{(z-1)^m}{m!} \langle X^m \rangle_f, \quad \langle X^m \rangle_f = \frac{d^m}{dz^m} G_X(z)\bigg|_{z=1}.$$

Function generating factorial cumulants:

$$\ln G_X(z) = \sum_{m=1}^{\infty} \frac{(z-1)^m}{m!} \langle\langle X^m \rangle\rangle_f, \quad \langle\langle X^m \rangle\rangle_f = \frac{d^m}{dz^m} \ln G_X(z)\bigg|_{z=1}.$$

Applications:

- ▷ Moments and cumulants of the Poisson distribution [nex16]
- ▷ Pascal distribution [nex22]
- ▷ Reconstructing probability distributions [nex14]

Multivariate Distributions [nln7]

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector variable with n components.

Joint probability distribution: $P(x_1, \ldots, x_n)$.

Marginal probability distribution:

$$P(x_1, \ldots, x_m) = \int dx_{m+1} \cdots dx_n \, P(x_1, \ldots, x_n).$$

Conditional probability distribution: $P(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n)$.

$$P(x_1, \ldots, x_n) = P(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n) P(x_{m+1}, \ldots, x_n).$$

Moments: $\langle X_1^{m_1} \cdots X_n^{m_n} \rangle = \int dx_1 \cdots dx_n \, x_1^{m_1} \cdots x_n^{m_n} P(x_1, \ldots, x_n)$.

Characteristic function: $\Phi(\mathbf{k}) = \langle e^{i\mathbf{k}\cdot\mathbf{X}} \rangle$.

Moment expansion: $\Phi(\mathbf{k}) = \displaystyle\sum_0^\infty \frac{(ik_1)^{m_1} \cdots (ik_n)^{m_n}}{m_1! \ldots m_n!} \langle X_1^{m_1} \cdots X_n^{m_n} \rangle$.

Cumulant expansion: $\ln \Phi(\mathbf{k}) = \displaystyle\sum_0^{\infty}{}' \frac{(ik_1)^{m_1} \cdots (ik_n)^{m_n}}{m_1! \ldots m_n!} \langle\langle X_1^{m_1} \cdots X_n^{m_n} \rangle\rangle$.

(prime indicates absence of term with $m_1 = \cdots = m_n = 0$).

Covariance matrix: $\langle\langle X_i X_j \rangle\rangle = \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle$.
($i = j$: variances, $i \neq j$: covariances).

Correlations: $C(X_i, X_j) = \dfrac{\langle\langle X_i X_j \rangle\rangle}{\sqrt{\langle\langle X_i \rangle\rangle \langle\langle X_j \rangle\rangle}}$.

Statistical independence of X_1, X_2: $P(x_1, x_2) = P_1(x_1) P_2(x_2)$.

Equivalent criteria for statistical independence:

- all moments factorize: $\langle X_1^{m_1} X_2^{m_2} \rangle = \langle X_1^{m_1} \rangle \langle X_2^{m_2} \rangle$;
- characteristic function factorizes: $\Phi(k_1, k_2) = \Phi_1(k_1) \Phi_2(k_2)$;
- all cumulants $\langle\langle X_1^{m_1} X_2^{m_2} \rangle\rangle$ with $m_1 m_2 \neq 0$ vanish.

If $\langle\langle X_1 X_2 \rangle\rangle = 0$ then X_1, X_2 are called *uncorrelated*.
This property does not imply *statistical independence*.

Transformation of Random Variables [nln49]

Consider two random variables X and Y that are functionally related:

$$Y = F(X) \quad \text{or} \quad X = G(Y).$$

If the probability distribution for X is known then the probability distribution for Y is determined as follows:

$$P_Y(y)\Delta y = \int_{y<f(x)<y+\Delta y} dx P_X(x)$$

$$\Rightarrow \; P_Y(y) = \int dx P_X(x)\delta\big(y - f(x)\big) = P_X\big(g(y)\big)\,|g'(y)|\,.$$

Consider two random variables X_1, X_2 with a joint probability distribution

$$P_{12}(x_1, x_2).$$

The probability distribution of the random variable $Y = X_1 + X_2$ is then determined as

$$P_Y(y) = \int dx_1 \int dx_2\, P_{12}(x_1, x_2)\delta(y - x_1 - x_2) = \int dx_1 P_{12}(x_1, y - x_1),$$

and the probability distribution of the random variable $Z = X_1X_2$ as

$$P_Z(z) = \int dx_1 \int dx_2\, P_{12}(x_1, x_2)\delta(z - x_1x_2) = \int \frac{dx_1}{|x_1|} P_{12}(x_1, z/x_1).$$

If the two random variables X_1, X_2 are statistically independent we can substitute $P_{12}(x_1, x_2) = P_1(x_1)P_2(x_2)$ in the above integrals.

Applications:

- ▷ Transformation of statistical uncertainty [nex24]
- ▷ Chebyshev inequality [nex6]
- ▷ Robust probability distributions [nex19]
- ▷ Statistically independent or merely uncorrelated? [nex23]
- ▷ Sum and product of uniform distributions [nex96]
- ▷ Exponential integral distribution [nex79]
- ▷ Generating exponential and Lorentzian random numbers [nex80]
- ▷ From Gaussian to exponential distribution [nex8]
- ▷ Transforming a pair of random variables [nex78]

[nex127] Sums of independent exponentials

Consider n independent random variable $X_1, \ldots, X_n$ with range $x_i \geq 0$ and identical exponential distributions,

$$P_1(x_i) = \frac{1}{\xi}\, e^{-x_i/\xi}, \quad i = 1, \ldots, n.$$

Use the transformation relation from [nln49],

$$P_2(x) = \int dx_1 \int dx_2\, P_1(x_1)P_1(x_2)\delta(x - x_1 - x_2) = \int dx_1 P_1(x_1)P_1(x - x_1),$$

inductively to calculate the probability distribution $P_n(x)$, $n \geq 2$ of the stochastic variable

$$X = X_1 + \cdots + X_n.$$

Find the mean value $\langle X \rangle$, the variance $\langle\langle X^2 \rangle\rangle$, and the value x_p where $P_n(x)$ has its peak value.

Solution:

[nex24] Transformation of statistical uncertainty.

From a given stochastic variable X with probability distribution $P_X(x)$ we can calculate the probability distribution of the stochastic variable $Y = f(X)$ via the relation

$$P_Y(y) = \int dx\, P_X(x)\delta\left(y - f(x)\right).$$

Show by systematic expansion that if $P_X(x)$ is sufficiently narrow and $f(x)$ sufficiently smooth, then the mean values and the standard deviations of the two stochastic variables are related to each other as follows:

$$\langle Y\rangle = f(\langle X\rangle), \qquad \sigma_Y = |f'(\langle X\rangle)|\sigma_X.$$

Solution:

[nex6] Chebyshev's inequality

Chebyshev's inequality is a rigorous relation between the standard deviation $\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$ of the random variable X and the probability of deviations from the mean value $\langle X \rangle$ greater than a given magnitude a.

$$P[(x - \langle X \rangle)^2 > a^2] \leq \left(\frac{\sigma_X}{a}\right)^2$$

Prove Chebyshev's inequality starting from the following relation, commonly used for the transformation of stochastic variables (as in [nln49]):

$$P_Y(y) = \int dx\, \delta(y - f(x)) P_X(x) \text{ with } f(x) = (x - \langle X \rangle)^2.$$

Solution:

[nex7] Law of large numbers

Let $X_1, \ldots, X_N$ be N statistically independent random variables described by the same probability distribution $P_X(x)$ with mean value $\langle X \rangle$ and standard deviation $\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$. These random variables might represent, for example, a series of measurements under the same (controllable) conditions. The law of large numbers states that the uncertainty (as measured by the standard deviation) of the stochastic variable $Y = (X_1 + \cdots + X_N)/N$ is

$$\sigma_Y = \frac{\sigma_X}{\sqrt{N}}.$$

Prove this result.

Solution:

Binomial, Poisson, and Gaussian Distributions [nln8]

Consider a set of N independent experiments, each having two possible outcomes occurring with given probabilities.

events	$A + B = S$
probabilities	$p + q = 1$
random variables	$n + m = N$

Binomial distribution:

$$P_N(n) = \frac{N!}{n!(N-n)!}\, p^n (1-p)^{N-n}.$$

Mean value: $\langle n \rangle = Np$.

Variance: $\langle\langle n^2 \rangle\rangle = Npq$. [nex15]]

In the following we consider two different asymptotic distributions in the limit $N \to \infty$.

Poisson distribution:

Limit #1: $N \to \infty$, $p \to 0$ such that $Np = \langle n \rangle = a$ stays finite [nex15].

$$P(n) = \frac{a^n}{n!}\, e^{-a}.$$

Cumulants: $\langle\langle n^m \rangle\rangle = a$.

Factorial cumulants: $\langle\langle n^m \rangle\rangle_f = a\delta_{m,1}$. [nex16]

Single parameter: $\langle n \rangle = \langle\langle n^2 \rangle\rangle = a$.

Gaussian distribution:

Limit #2: $N \gg 1$, $p > 0$ with $Np \gg \sqrt{Npq}$.

$$P_N(n) = \frac{1}{\sqrt{2\pi\langle\langle n^2 \rangle\rangle}} \exp\left(-\frac{(n - \langle n \rangle)^2}{2\langle\langle n^2 \rangle\rangle}\right).$$

Derivation: DeMoivre-Laplace limit theorem [nex21].

Two parameters: $\langle n \rangle = Np$, $\langle\langle n^2 \rangle\rangle = Npq$.

Special case of central limit theorem [nln9].

[nex15] Binomial to Poisson distribution

Consider the binomial distribution for two events A, B that occur with probabilities $P(A) \equiv p$, $P(B) = 1 - p \equiv q$, respectively:

$$P_N(n) = \frac{N!}{n!(N-n)!}\, p^n q^{N-n},$$

where N is the number of (independent) experiments performed, and n is the stochastic variable that counts the number of realizations of event A.
(a) Find the mean value $\langle n \rangle$ and the variance $\langle\langle n^2 \rangle\rangle$ of the stochastic variable n.
(b) Show that for $N \to \infty$, $p \to 0$ with $Np \to a > 0$, the binomial distribution turns into the Poisson distribution

$$P_\infty(n) = \frac{a^n}{n!}\, e^{-a}.$$

Solution:

[nex21] De Moivre–Laplace limit theorem.

Show that for large Np and large Npq the binomial distribution turns into the Gaussian distribution with the same mean value $\langle n\rangle = Np$ and variance $\langle\langle n^2\rangle\rangle = Npq$:

$$P_N(n) = \frac{N!}{n!(N-n)!}\, p^n q^{N-n} \quad \longrightarrow \quad P_N(n) \simeq \frac{1}{\sqrt{2\pi\langle\langle n^2\rangle\rangle}} \exp\left(-\frac{(n-\langle n\rangle)^2}{2\langle\langle n^2\rangle\rangle}\right).$$

Solution:

Central Limit Theorem [nln9]

The central limit theorem is a major extension of the law of large numbers. It explains the unique role of the Gaussian distribution in statistical physics.

Given are a large number of statistically independent random variables $X_i, i = 1, \ldots, N$ with equal probability distributions $P_X(x_i)$. The only restriction on the shape of $P_X(x_i)$ is that the moments $\langle X_i^n \rangle = \langle X^n \rangle$ are finite for all n.

Goal: Find the probability distribution $P_Y(y)$ for the random variable $Y = (X_1 - \langle X \rangle + \cdots + X_N - \langle X \rangle)/N$.

$$P_Y(y) = \int dx_1\, P_X(x_1) \cdots \int dx_N\, P_X(x_N) \delta\left(y - \frac{1}{N}\sum_{i=1}^{N} [x_i - \langle X \rangle] \right).$$

Characteristic function:

$$\Phi_Y(k) \equiv \int dy\, e^{iky} P_Y(y), \quad P_Y(y) = \frac{1}{2\pi}\int dk\, e^{-iky}\Phi_Y(k).$$

$$\begin{aligned}
\Rightarrow\ \Phi_Y(k) &= \int dx_1\, P_X(x_1) \cdots \int dx_N\, P_X(x_N) \exp\left(i\frac{k}{N}\sum_{i=1}^{N} [x_i - \langle X \rangle] \right) \\
&= \left[\bar{\Phi}\left(k/N\right)\right]^N,
\end{aligned}$$

$$\begin{aligned}
\bar{\Phi}\left(\frac{k}{N}\right) &= \int dx\, e^{i(k/N)(x - \langle X \rangle)} P_X(x) = \exp\left(-\frac{1}{2}\left(\frac{k}{N}\right)^2 \langle\langle X^2 \rangle\rangle + \cdots \right) \\
&= 1 - \frac{1}{2}\left(\frac{k}{N}\right)^2 \langle\langle X^2 \rangle\rangle + \mathrm{O}\left(\frac{k^3}{N^3}\right),
\end{aligned}$$

where we have performed a cumulant expansion to leading order.

$$\Rightarrow\ \Phi_Y(y) = \left[1 - \frac{k^2 \langle\langle X^2 \rangle\rangle}{2N^2} + \mathrm{O}\left(\frac{k^3}{N^3}\right) \right]^N \stackrel{N\to\infty}{\longrightarrow} \exp\left(-\frac{k^2 \langle\langle X^2 \rangle\rangle}{2N} \right).$$

where we have used $\lim_{N\to\infty}(1 + z/N)^N = e^z$.

$$\Rightarrow\ P_Y(y) = \sqrt{\frac{N}{2\pi\langle\langle X^2 \rangle\rangle}} \exp\left(-\frac{Ny^2}{2\langle\langle X^2 \rangle\rangle} \right) = \frac{1}{\sqrt{2\pi\langle\langle Y^2 \rangle\rangle}}\, e^{-y^2/2\langle\langle Y^2 \rangle\rangle}$$

with variance $\langle\langle Y^2 \rangle\rangle = \langle\langle X^2 \rangle\rangle / N$

Note that regardless of the form of $P_X(x)$, the average of a large number of (independent) measurements of X will be a Gaussian with standard deviation $\sigma_Y = \sigma_X/\sqrt{N}$.

[nex19] Robust probability distributions

Consider two independent stochastic variables X_1 and X_2, each specified by the same probability distribution $P_X(x)$. Show that if $P_X(x)$ is either a Gaussian, a Lorentzian, or a Poisson distribution,

$$\text{(i)}\quad P_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-x^2/2\sigma^2}, \qquad \text{(ii)}\quad P_X(x) = \frac{1}{\pi}\frac{a}{x^2+a^2}, \qquad \text{(iii)}\quad P_X(x=n) = \frac{a^n}{n!}\, e^{-a}.$$

then the probability distribution $P_Y(y)$ of the stochastic variable $Y = X_1 + X_2$ is also a Gaussian, a Lorentzian, or a Poisson distribution, respectively. What property of the characteristic function $\Phi_X(k)$ guarantees the robustness of $P_X(x)$?

Solution:

[nex81] Stable probability distributions

Consider N independent random variables $X_1, \ldots, X_N$, each having the same probability distribution $P_X(x)$. If the probability distribution of the random variable $Y_N = X_1 + \cdots + X_N$ can be written in the form $P_Y(y) = P_X(y/c_N + \gamma_N)/c_N$, then $P_X(x)$ is *stable*. The multiplicative constant must be of the form $c_N = N^{1/\alpha}$, where α is the *index* of the stable distribution. $P_X(x)$ is *strictly stable* if $\gamma_N = 0$.
Use the results of [nex19] to determine the indices α of the Gaussian and Lorentzian distributions, both of which are both strictly stable. Show that the Poisson distribution is not stable in the technical sense used here.

Solution:

Exponential distribution [nln10]

Busses arrive randomly at a bus station.
The average interval between successive bus arrivals is τ.

$f(t)dt$: probability that the interval is between t and $t+dt$.

$P_0(t) = \int_t^\infty dt' f(t')$: probability that the interval is larger than t.

Relation: $f(t) = -\frac{dP_0}{dt}$.

Normalizations: $P_0(0) = 1, \quad \int_0^\infty dt\, f(t) = 1$.

Mean value: $\langle t \rangle \equiv \int_0^\infty dt\, t f(t) = \tau$.

Start the clock when a bus has arrived and consider the events A and B.

Event A: the next bus has not arrived by time t.
Event B: a bus arrives between times t and $t+dt$.

Assumptions:

1. $P(AB) = P(A)P(B)$ (statistical independence).
2. $P(B) = cdt$ with c to be determined.

Consequence: $P_0(t+dt) = P(A\bar{B}) = P(A)P(\bar{B}) = P_0(t)[1 - cdt]$.

$$\Rightarrow \frac{d}{dt}P_0(t) = -cP_0(t) \Rightarrow P_0(t) = e^{-ct} \Rightarrow f(t) = ce^{-ct}.$$

Adjust mean value: $\langle t \rangle = \tau \Rightarrow c = 1/\tau$.

Exponential distribution: $P_0(t) = e^{-t/\tau}, \quad f(t) = \frac{1}{\tau}e^{-t/\tau}$.

Find the probability $P_n(t)$ that n busses arrive before time t.

First consider the probabilities $f(t')dt'$ and $P_0(t-t')$ of the two statistically independent events that the first bus arrives between t' and $t'+dt'$ and that no futher bus arrives until time t.

Probability that exactly one bus arrives until time t:

$$P_1(t) = \int_0^t dt' f(t')P_0(t-t') = \frac{t}{\tau}e^{-t/\tau}.$$

Then calculate $P_n(t)$ by induction.

Poisson distribution: $P_n(t) = \int_0^t dt' f(t')P_{n-1}(t-t') = \frac{(t/\tau)^n}{n!}e^{-t/\tau}$.

Waiting Time Problem [nln11]

Busses arrive more or less randomly at a bus station.
Given is the probability distribution $f(t)$ for intervals between bus arrivals.

Normalization: $\displaystyle\int_0^\infty dt\, f(t) = 1$.

Probability that the interval is larger than t: $\displaystyle P_0(t) = \int_t^\infty dt' f(t')$.

Mean time interval between arrivals: $\displaystyle\tau_B = \int_0^\infty dt\, t f(t) = \int_0^\infty dt P_0(t)$.

Find the probability $Q_0(t)$ that no arrivals occur in a randomly chosen time interval of length t.

First consider the probability $P_0(t'+t)$ for this to be the case if the interval starts at time t' after the last bus arrival. Then average $P_0(t'+t)$ over the range of elapsed time t'.

$$\Rightarrow\; Q_0(t) = c\int_0^\infty dt' P_0(t'+t) \text{ with normalization } Q_0(0) = 1.$$

$$\Rightarrow\; Q_0(t) = \frac{1}{\tau_B}\int_t^\infty dt' P_0(t').$$

Passengers come to the station at random times. The probability that a passenger has to wait at least a time t before the next bus is then $Q_0(t)$:

Probabilty distribution of passenger waiting times:

$$g(t) = -\frac{d}{dt}Q_0(t) = \frac{1}{\tau_B}P_0(t).$$

Mean passenger waiting time: $\displaystyle\tau_P = \int_0^\infty dt\, t g(t) = \int_0^\infty dt Q_0(t)$.

The relationship between τ_B and τ_P depends on the distribution $f(t)$. In general, we have $\tau_P \leq \tau_B$. The equality sign holds for the exponential distribution.

[nex22] Pascal distribution.

Consider the quantum harmonic oscillator in thermal equilibrium at temperature T. The energy levels (relative to the ground state) are $E_n = n\hbar\omega$, $n = 0, 1, 2, \ldots$
(a) Show that the system is in level n with probability

$$P(n) = (1-\gamma)\gamma^n, \quad \gamma = \exp(-\hbar\omega/k_BT).$$

$P(n)$ is called *Pascal* distribution or *geometric* distribution.
(b) Calculate the factorial moments $\langle n^m\rangle_f$ and the factorial cumulants $\langle\langle n^m\rangle\rangle_f$ of this distribution.
(c) Show that the Pascal distribution has a larger variance $\langle\langle n^2\rangle\rangle$ than the Poisson distribution with the same mean value $\langle n\rangle$.

Solution: