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Probability Distribution

Experiment represented by events in a sample space: \( S = \{ A_1, A_2, \ldots \} \).
Measurements represented by stochastic variable: \( X = \{ x_1, x_2, \ldots \} \).
Maximum amount of information experimentally obtainable is contained in the probability distribution:
\[
P_X(x_i) \geq 0, \quad \sum_i P_X(x_i) = 1.
\]
Partial information is contained in moments,
\[
\langle X^n \rangle = \sum_i x_i^n P_X(x_i), \quad n = 1, 2, \ldots,
\]
or cumulants (as defined in [ln47]),
\[
\begin{align*}
\langle \langle X \rangle \rangle &= \langle X \rangle \quad \text{(mean value)} \\
\langle \langle X^2 \rangle \rangle &= \langle X^2 \rangle - \langle X \rangle^2 \quad \text{(variance)} \\
\langle \langle X^3 \rangle \rangle &= \langle X^3 \rangle - 3\langle X \rangle \langle X^2 \rangle + 2\langle X \rangle^3
\end{align*}
\]
The variance is the square of the standard deviation: \( \langle \langle X^2 \rangle \rangle = \sigma_X^2 \).

For continuous stochastic variables we have
\[
P_X(x) \geq 0, \quad \int dx P_X(x) = 1, \quad \langle X^n \rangle = \int dx x^n P(x).
\]
In the literature \( P_X(x) \) is often named ‘probability density’ and the term ‘distribution’ is used for
\[
F_X(x) = \sum_{x_i < x} P_X(x_i) \quad \text{or} \quad F_X(x) = \int_{-\infty}^{x} dx' P_X(x')
\]
in a cumulative sense.
Characteristic Function

Fourier transform: \( \Phi_X(k) \doteq \langle e^{ikx} \rangle = \int_{-\infty}^{+\infty} dx \, e^{ikx} P_X(x) \).

Attributes: \( \Phi_X(0) = 1 \), \( |\Phi_X(k)| \leq 1 \).

Moment generating function:
\[
\Phi_X(k) = \int_{-\infty}^{+\infty} dx \, P_X(x) \left( \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} x^n \right) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle X^n \rangle
\]
\[
\Rightarrow \langle X^n \rangle = \int_{-\infty}^{+\infty} dx \, x^n P_X(x) = (-i)^n \frac{d^n}{dk^n} \Phi_X(k) \bigg|_{k=0}.
\]

Cumulant generating function:
\[
\ln \Phi_X(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle \langle X^n \rangle \rangle \Rightarrow \langle \langle X^n \rangle \rangle = (-i)^n \frac{d^n}{dk^n} \ln \Phi_X(k) \bigg|_{k=0}.
\]

Cumulants in terms of moments (with \( \Delta X \doteq X - \langle X \rangle \)): [nex126]

- \( \langle \langle X \rangle \rangle = \langle X \rangle \)
- \( \langle \langle X^2 \rangle \rangle = \langle X^2 \rangle - \langle X \rangle^2 = \langle (\Delta X)^2 \rangle \)
- \( \langle \langle X^3 \rangle \rangle = \langle (\Delta X)^3 \rangle \)
- \( \langle \langle X^4 \rangle \rangle = \langle (\Delta X)^4 \rangle - 3 \langle (\Delta X)^2 \rangle^2 \)

Theorem of Marcienkiewicz: \( \ln \Phi_X(k) \) can only be a polynomial if the degree is \( n \leq 2 \).

- \( n = 1: \ \ln \Phi_X(k) = ika \Rightarrow P_X(x) = \delta(x - a) \)
- \( n = 2: \ \ln \Phi_X(k) = ika - \frac{1}{2} bk^2 \Rightarrow P_X(x) = \frac{1}{\sqrt{2\pi b}} \exp \left( -\frac{(x-a)^2}{2b} \right) \)

Consequence: any probability distribution has either one, two, or infinitely many non-vanishing cumulants.
Cumulants expressed in terms of moments

The characteristic function $\Phi_X(k)$ of a probability distribution $P_X(x)$, obtained via Fourier transform as described in [nlh47], can be used to generate the moments $\langle X^n \rangle$ and the cumulants $\langle\langle X^n \rangle\rangle$ via the expansions

$$
\Phi_X(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle X^n \rangle,
\ln \Phi_X(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle\langle X^n \rangle\rangle.
$$

Use these relations to express the first four cumulants in terms of the first four moments. The results are stated in [nlh47]. Describe your work in some detail.

Solution:
Generating function

The generating function $G_X(z)$ is a representation of the characteristic function $\Phi_X(k)$ that is most commonly used, along with factorial moments and factorial cumulants, if the stochastic variable $X$ is integer valued.

Definition: $G_X(z) \doteq \langle z^X \rangle$ with $|z| = 1$.

Application to continuous and discrete (integer-valued) stochastic variables:

$$G_X(z) = \int dx \, z^x P_X(x), \quad G_X(z) = \sum_n z^n P_X(n).$$

Definition of factorial moments:

$$\langle X^m \rangle_f \doteq \langle (X-1) \cdots (X-m+1) \rangle, \quad m \geq 1; \quad \langle X^0 \rangle_f \doteq 0.$$

Function generating factorial moments:

$$G_X(z) = \sum_{m=0}^{\infty} \frac{(z-1)^m}{m!} \langle X^m \rangle_f, \quad \langle X^m \rangle_f = \frac{d^m}{dz^m} G_X(z) \bigg|_{z=1}.$$

Function generating factorial cumulants:

$$\ln G_X(z) = \sum_{m=1}^{\infty} \frac{(z-1)^m}{m!} \langle \langle X^m \rangle \rangle_f, \quad \langle \langle X^m \rangle \rangle_f = \frac{d^m}{dz^m} \ln G_X(z) \bigg|_{z=1}.$$

Applications:

▷ Moments and cumulants of the Poisson distribution [nex16]
▷ Pascal distribution [nex22]
▷ Reconstructing probability distributions [nex14]
Multivariate Distributions

Let \( X = (X_1, \ldots, X_n) \) be a random vector variable with \( n \) components.

Joint probability distribution: \( P(x_1, \ldots, x_n) \).

Marginal probability distribution:

\[
P(x_1, \ldots, x_m) = \int dx_{m+1} \cdots dx_n P(x_1, \ldots, x_n).
\]

Conditional probability distribution:

\[
P(x_1, \ldots, x_n) = P(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n) P(x_{m+1}, \ldots, x_n).
\]

Moments:

\[
\langle X_1^{m_1} \cdots X_n^{m_n} \rangle = \int dx_1 \cdots dx_n x_1^{m_1} \cdots x_n^{m_n} P(x_1, \ldots, x_n).
\]

Characteristic function:

\[
\Phi(k) = \langle e^{i k \cdot X} \rangle.
\]

Moment expansion:

\[
\Phi(k) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(ik_1)^{m_1} \cdots (ik_n)^{m_n}}{m_1! \cdots m_n!} \langle X_1^{m_1} \cdots X_n^{m_n} \rangle.
\]

Cumulant expansion:

\[
\ln \Phi(k) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(ik_1)^{m_1} \cdots (ik_n)^{m_n}}{m_1! \cdots m_n!} \langle \langle X_1^{m_1} \cdots X_n^{m_n} \rangle \rangle.
\]

(prime indicates absence of term with \( m_1 = \cdots = m_n = 0 \)).

Covariance matrix:

\[
\langle X_i X_j \rangle = \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle.
\]

\( (i = j: \text{variances} , \ i \neq j: \text{covariances}) \).

Correlations:

\[
C(X_i, X_j) = \frac{\langle X_i X_j \rangle}{\sqrt{\langle X_i^2 \rangle \langle X_j^2 \rangle}}.
\]

Statistical independence of \( X_1, X_2 \):

\( P(x_1, x_2) = P_1(x_1)P_2(x_2) \).

Equivalent criteria for statistical independence:

\begin{itemize}
  
  \item all moments factorize: \( \langle X_1^{m_1} X_2^{m_2} \rangle = \langle X_1^{m_1} \rangle \langle X_2^{m_2} \rangle \);
  
  \item characteristic function factorizes: \( \Phi(k_1, k_2) = \Phi_1(k_1)\Phi_2(k_2) \);
  
  \item all cumulants \( \langle \langle X_1^{m_1} X_2^{m_2} \rangle \rangle \) with \( m_1 m_2 \neq 0 \) vanish.
\end{itemize}

If \( \langle X_1 X_2 \rangle = 0 \) then \( X_1, X_2 \) are called \textit{uncorrelated}.

This property does not imply \textit{statistical independence}. 
Consider two random variables $X$ and $Y$ that are functionally related:

$$Y = F(X) \quad \text{or} \quad X = G(Y).$$

If the probability distribution for $X$ is known then the probability distribution for $Y$ is determined as follows:

$$P_Y(y) \Delta y = \int_{y < f(x) < y + \Delta y} dx P_X(x)$$

$$\Rightarrow P_Y(y) = \int dx P_X(x) \delta(y - f(x)) = P_X(g(y)) |g'(y)|.$$

Consider two random variables $X_1, X_2$ with a joint probability distribution $P_{12}(x_1, x_2)$.

The probability distribution of the random variable $Y = X_1 + X_2$ is then determined as

$$P_Y(y) = \int dx_1 \int dx_2 P_{12}(x_1, x_2) \delta(y - x_1 - x_2) = \int dx_1 P_{12}(x_1, y - x_1),$$

and the probability distribution of the random variable $Z = X_1 X_2$ as

$$P_Z(z) = \int dx_1 \int dx_2 P_{12}(x_1, x_2) \delta(z - x_1 x_2) = \int \frac{dx_1}{|x_1|} P_{12}(x_1, z/x_1).$$

If the two random variables $X_1, X_2$ are statistically independent we can substitute $P_{12}(x_1, x_2) = P_1(x_1)P_2(x_2)$ in the above integrals.

Applications:

- Transformation of statistical uncertainty [nex24]
- Chebyshev inequality [nex6]
- Robust probability distributions [nex19]
- Statistically independent or merely uncorrelated? [nex23]
- Sum and product of uniform distributions [nex96]
- Exponential integral distribution [nex79]
- Generating exponential and Lorentzian random numbers [nex80]
- From Gaussian to exponential distribution [nex8]
- Transforming a pair of random variables [nex78]
Consider $n$ independent random variables $X_1, \ldots, X_n$ with range $x_i \geq 0$ and identical exponential distributions,

$$P_1(x_i) = \frac{1}{\xi} e^{-x_i/\xi}, \quad i = 1, \ldots, n.$$ 

Use the transformation relation from [hn-49],

$$P_2(x) = \int dx_1 \int dx_2 P_1(x_1)P_1(x_2)\delta(x - x_1 - x_2) = \int dx_1 P_1(x_1)P_1(x - x_1),$$

inductively to calculate the probability distribution $P_n(x), n \geq 2$ of the stochastic variable

$$X = X_1 + \cdots + X_n.$$

Find the mean value $\langle X \rangle$, the variance $\langle X^2 \rangle$, and the value $x_p$ where $P_n(x)$ has its peak value.

**Solution:**

From a given stochastic variable $X$ with probability distribution $P_X(x)$ we can calculate the probability distribution of the stochastic variable $Y = f(X)$ via the relation

$$P_Y(y) = \int dx P_X(x) \delta (y - f(x)).$$

Show by systematic expansion that if $P_X(x)$ is sufficiently narrow and $f(x)$ sufficiently smooth, then the mean values and the standard deviations of the two stochastic variables are related to each other as follows:

$$\langle Y \rangle = f(\langle X \rangle), \quad \sigma_Y = |f'(\langle X \rangle)|\sigma_X.$$

Solution:
[nex6] Chebyshev’s inequality

Chebyshev’s inequality is a rigorous relation between the standard deviation $\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$ of the random variable $X$ and the probability of deviations from the mean value $\langle X \rangle$ greater than a given magnitude $a$.

$$P[(x - \langle X \rangle)^2 > a^2] \leq \left( \frac{\sigma_X}{a} \right)^2$$

Prove Chebyshev’s inequality starting from the following relation, commonly used for the transformation of stochastic variables (as in [nln49]):

$$P_Y(y) = \int dx \delta(y - f(x)) P_X(x) \text{ with } f(x) = (x - \langle X \rangle)^2.$$

Solution:
Law of large numbers

Let $X_1, \ldots, X_N$ be $N$ statistically independent random variables described by the same probability distribution $P_X(x)$ with mean value $\langle X \rangle$ and standard deviation $\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$. These random variables might represent, for example, a series of measurements under the same (controllable) conditions. The law of large numbers states that the uncertainty (as measured by the standard deviation) of the stochastic variable $Y = (X_1 + \cdots + X_N)/N$ is

$$\sigma_Y = \frac{\sigma_X}{\sqrt{N}}.$$

Prove this result.

Solution:
Consider a set of $N$ independent experiments, each having two possible outcomes occurring with given probabilities.

\begin{align*}
\text{events} & \quad A + B = S \\
\text{probabilities} & \quad p + q = 1 \\
\text{random variables} & \quad n + m = N
\end{align*}

**Binomial distribution:**

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.$$  

- Mean value: $\langle n \rangle = N p$.
- Variance: $\langle \langle n^2 \rangle \rangle = Npq$.  \[nex15\]

In the following we consider two different asymptotic distributions in the limit $N \to \infty$.

**Poisson distribution:**

Limit #1: $N \to \infty$, $p \to 0$ such that $Np = \langle n \rangle = a$ stays finite \[nex15\].

$$P(n) = \frac{a^n}{n!} e^{-a}.$$  

- Cumulants: $\langle \langle n^m \rangle \rangle = a$.
- Factorial cumulants: $\langle \langle n^m \rangle \rangle_f = a \delta_{m,1}$. \[nex16\]
- Single parameter: $\langle n \rangle = \langle \langle n^2 \rangle \rangle = a$.

**Gaussian distribution:**

Limit #2: $N \gg 1$, $p > 0$ with $Np \gg \sqrt{Npq}$.

$$P_N(n) = \frac{1}{\sqrt{2\pi\langle \langle n^2 \rangle \rangle}} \exp \left( -\frac{(n - \langle n \rangle)^2}{2\langle \langle n^2 \rangle \rangle} \right).$$  

- Derivation: DeMoivre-Laplace limit theorem \[nex21\].
- Two parameters: $\langle n \rangle = Np$, $\langle \langle n^2 \rangle \rangle = Npq$.
- Special case of central limit theorem \[nlh9\].
Binomial to Poisson distribution

Consider the binomial distribution for two events $A, B$ that occur with probabilities $P(A) \equiv p$, $P(B) = 1 - p \equiv q$, respectively:

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n},$$

where $N$ is the number of (independent) experiments performed, and $n$ is the stochastic variable that counts the number of realizations of event $A$.

(a) Find the mean value $\langle n \rangle$ and the variance $\langle n^2 \rangle$ of the stochastic variable $n$.

(b) Show that for $N \to \infty$, $p \to 0$ with $Np \to a > 0$, the binomial distribution turns into the Poisson distribution

$$P_\infty(n) = \frac{a^n}{n!} e^{-a}.$$

Solution:
De Moivre–Laplace limit theorem.

Show that for large $Np$ and large $Npq$ the binomial distribution turns into the Gaussian distribution with the same mean value $\langle n \rangle = Np$ and variance $\langle \langle n^2 \rangle \rangle = Npq$:

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n} \quad \longrightarrow \quad P_N(n) \simeq \frac{1}{\sqrt{2\pi\langle n^2 \rangle}} \exp \left( -\frac{(n - \langle n \rangle)^2}{2\langle n^2 \rangle} \right).$$

Solution:
Central Limit Theorem

The central limit theorem is a major extension of the law of large numbers. It explains the unique role of the Gaussian distribution in statistical physics.

Given are a large number of statistically independent random variables $X_i, i = 1, \ldots, N$ with equal probability distributions $P_X(x_i)$. The only restriction on the shape of $P_X(x_i)$ is that the moments $\langle X_i^n \rangle = \langle X^n \rangle$ are finite for all $n$.

Goal: Find the probability distribution $P_Y(y)$ for the random variable $Y = (X_1 - \langle X \rangle + \cdots + X_N - \langle X \rangle)/N$.

$$ P_Y(y) = \int dx_1 P_X(x_1) \cdots \int dx_N P_X(x_N) \delta \left( y - \frac{1}{N} \sum_{i=1}^N [x_i - \langle X \rangle] \right). $$

Characteristic function:

$$ \Phi_Y(k) \equiv \int dy e^{iky} P_Y(y), \quad P_Y(y) = \frac{1}{2\pi} \int dk e^{-iky} \Phi_Y(k). $$

$$ \Rightarrow \Phi_Y(k) = \int dx_1 P_X(x_1) \cdots \int dx_N P_X(x_N) \exp \left( i \frac{k}{N} \sum_{i=1}^N [x_i - \langle X \rangle] \right) $$

$$ = \left[ \Phi \left( \frac{k}{N} \right) \right]^N, $$

$$ \Phi \left( \frac{k}{N} \right) = \int dx e^{i(k/N)(x-\langle X \rangle)} P_X(x) = \exp \left( -\frac{1}{2} \left( \frac{k}{N} \right)^2 \langle \langle X^2 \rangle \rangle + \cdots \right) $$

$$ = 1 - \frac{1}{2} \left( \frac{k}{N} \right)^2 \langle \langle X^2 \rangle \rangle + O \left( \frac{k^3}{N^3} \right), $$

where we have performed a cumulant expansion to leading order.

$$ \Rightarrow \Phi_Y(y) = \left[ 1 - \frac{k^2}{2N^2} + O \left( \frac{k^3}{N^3} \right) \right]^N \xrightarrow{N \to \infty} \exp \left( -\frac{k^2}{2N} \langle \langle Y^2 \rangle \rangle \right). $$

where we have used $\lim_{N \to \infty} (1 + z/N)^N = e^z$.

$$ \Rightarrow P_Y(y) = \sqrt{\frac{N}{2\pi \langle \langle Y^2 \rangle \rangle}} \exp \left( -\frac{Ny^2}{2\langle \langle Y^2 \rangle \rangle} \right) = \frac{1}{\sqrt{2\pi \langle \langle Y^2 \rangle \rangle}} e^{-y^2/2\langle \langle Y^2 \rangle \rangle} $$

with variance $\langle \langle Y^2 \rangle \rangle = \langle \langle X^2 \rangle \rangle / N$

Note that regardless of the form of $P_X(x)$, the average of a large number of (independent) measurements of $X$ will be a Gaussian with standard deviation $\sigma_Y = \sigma_X / \sqrt{N}$. 
Consider two independent stochastic variables $X_1$ and $X_2$, each specified by the same probability distribution $P_X(x)$. Show that if $P_X(x)$ is either a Gaussian, a Lorentzian, or a Poisson distribution,

(i) $P_X(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-x^2/2\sigma^2}$,  
(ii) $P_X(x) = \frac{1}{\pi} \frac{a}{x^2 + a^2}$,  
(iii) $P_X(x = n) = \frac{a^n}{n!} e^{-a}$.

then the probability distribution $P_Y(y)$ of the stochastic variable $Y = X_1 + X_2$ is also a Gaussian, a Lorentzian, or a Poisson distribution, respectively. What property of the characteristic function $\Phi_X(k)$ guarantees the robustness of $P_X(x)$?

Solution:
[nex81] Stable probability distributions

Consider $N$ independent random variables $X_1, \ldots, X_N$, each having the same probability distribution $P_X(x)$. If the probability distribution of the random variable $Y_N = X_1 + \cdots + X_N$ can be written in the form $P_Y(y) = P_X(y/c_N + \gamma_N)/c_N$, then $P_X(x)$ is stable. The multiplicative constant must be of the form $c_N = N^{1/\alpha}$, where $\alpha$ is the index of the stable distribution. $P_X(x)$ is strictly stable if $\gamma_N = 0$.

Use the results of [nex19] to determine the indices $\alpha$ of the Gaussian and Lorentzian distributions, both of which are both strictly stable. Show that the Poisson distribution is not stable in the technical sense used here.

Solution:
Exponential distribution

Busses arrive randomly at a bus station.
The average interval between successive bus arrivals is $\tau$.

$f(t)dt$: probability that the interval is between $t$ and $t + dt$.

$P_0(t) = \int_t^\infty dt' f(t')$: probability that the interval is larger than $t$.

Relation: $f(t) = -\frac{dP_0}{dt}$.

Normalizations: $P_0(0) = 1$, $\int_0^\infty dt f(t) = 1$.

Mean value: $\langle t \rangle \equiv \int_0^\infty dt t f(t) = \tau$.

Start the clock when a bus has arrived and consider the events $A$ and $B$.

Event $A$: the next bus has not arrived by time $t$.
Event $B$: a bus arrives between times $t$ and $t + dt$.

Assumptions:

1. $P(AB) = P(A)P(B)$ (statistical independence).
2. $P(B) = cd t$ with $c$ to be determined.

Consequence: $P_0(t + dt) = P(A\bar{B}) = P(A)P(\bar{B}) = P_0(t)[1 - cd t]$.

$\Rightarrow \frac{d}{dt} P_0(t) = -cP_0(t) \Rightarrow P_0(t) = e^{-ct} \Rightarrow f(t) = ce^{-ct}$.

Adjust mean value: $\langle t \rangle = \tau \Rightarrow c = 1/\tau$.

Exponential distribution: $P_0(t) = e^{-t/\tau}$, $f(t) = \frac{1}{\tau} e^{-t/\tau}$.

Find the probability $P_n(t)$ that $n$ busses arrive before time $t$.

First consider the probabilities $f(t')dt'$ and $P_0(t - t')$ of the two statistically independent events that the first bus arrives between $t'$ and $t' + dt'$ and that no further bus arrives until time $t$.

Probability that exactly one bus arrives until time $t$:

$P_1(t) = \int_0^t dt' f(t') P_0(t - t') = \frac{t}{\tau} e^{-t/\tau}$.

Then calculate $P_n(t)$ by induction.

Poisson distribution: $P_n(t) = \int_0^t dt' f(t') P_{n-1}(t - t') = \frac{(t/\tau)^n}{n!} e^{-t/\tau}$. 
Waiting Time Problem

Busses arrive more or less randomly at a bus station. Given is the probability distribution \(f(t)\) for intervals between bus arrivals.

Normalization: \(\int_0^\infty dt f(t) = 1\).

Probability that the interval is larger than \(t\): \(P_0(t) = \int_t^\infty dt' f(t')\).

Mean time interval between arrivals: \(\tau_B = \int_0^\infty dt t f(t) = \int_0^\infty dt P_0(t)\).

Find the probability \(Q_0(t)\) that no arrivals occur in a randomly chosen time interval of length \(t\).

First consider the probability \(P_0(t' + t)\) for this to be the case if the interval starts at time \(t'\) after the last bus arrival. Then average \(P_0(t' + t)\) over the range of elapsed time \(t'\).

\[
\Rightarrow Q_0(t) = c \int_0^\infty dt' P_0(t' + t) \text{ with normalization } Q_0(0) = 1.
\]

\[
\Rightarrow Q_0(t) = \frac{1}{\tau_B} \int_t^\infty dt' P_0(t').
\]

Passengers come to the station at random times. The probability that a passenger has to wait at least a time \(t\) before the next bus is then \(Q_0(t)\):

Probabilty distribution of passenger waiting times:

\[
g(t) = -\frac{d}{dt} Q_0(t) = \frac{1}{\tau_B} P_0(t).
\]

Mean passenger waiting time: \(\tau_P = \int_0^\infty dt t g(t) = \int_0^\infty dt Q_0(t)\).

The relationship between \(\tau_B\) and \(\tau_P\) depends on the distribution \(f(t)\). In general, we have \(\tau_P \leq \tau_B\). The equality sign holds for the exponential distribution.
Pascal distribution.

Consider the quantum harmonic oscillator in thermal equilibrium at temperature $T$. The energy levels (relative to the ground state) are $E_n = n\hbar\omega$, $n = 0, 1, 2, \ldots$

(a) Show that the system is in level $n$ with probability

$$P(n) = (1 - \gamma)\gamma^n, \quad \gamma = \exp(-\hbar\omega/k_B T).$$

$P(n)$ is called Pascal distribution or geometric distribution.

(b) Calculate the factorial moments $\langle n^m \rangle_f$ and the factorial cumulants $\langle\langle n^m \rangle\rangle_f$ of this distribution.

(c) Show that the Pascal distribution has a larger variance $\langle\langle n^2 \rangle\rangle$ than the Poisson distribution with the same mean value $\langle n \rangle$.

Solution: