Exponential distribution [nln10]

Busses arrive randomly at a bus station.
The average interval between successive bus arrivals is τ.

$f(t)dt$: probability that the interval is between t and $t+dt$.

$P_0(t) = \int_t^\infty dt' f(t')$: probability that the interval is larger than t.

Relation: $f(t) = -\frac{dP_0}{dt}$.

Normalizations: $P_0(0) = 1, \quad \int_0^\infty dt\, f(t) = 1$.

Mean value: $\langle t \rangle \equiv \int_0^\infty dt\, t f(t) = \tau$.

Start the clock when a bus has arrived and consider the events A and B.

Event A: the next bus has not arrived by time t.
Event B: a bus arrives between times t and $t+dt$.

Assumptions:

1. $P(AB) = P(A)P(B)$ (statistical independence).
2. $P(B) = cdt$ with c to be determined.

Consequence: $P_0(t+dt) = P(A\bar{B}) = P(A)P(\bar{B}) = P_0(t)[1-cdt]$.

$$\Rightarrow \frac{d}{dt}P_0(t) = -cP_0(t) \;\Rightarrow\; P_0(t) = e^{-ct} \;\Rightarrow\; f(t) = ce^{-ct}.$$

Adjust mean value: $\langle t \rangle = \tau \;\Rightarrow\; c = 1/\tau$.

Exponential distribution: $P_0(t) = e^{-t/\tau}, \quad f(t) = \frac{1}{\tau}e^{-t/\tau}$.

Find the probability $P_n(t)$ that n busses arrive before time t.

First consider the probabilities $f(t')dt'$ and $P_0(t-t')$ of the two statistically independent events that the first bus arrives between t' and $t'+dt'$ and that no futher bus arrives until time t.

Probability that exactly one bus arrives until time t:

$$P_1(t) = \int_0^t dt' f(t') P_0(t-t') = \frac{t}{\tau}e^{-t/\tau}.$$

Then calculate $P_n(t)$ by induction.

Poisson distribution: $P_n(t) = \int_0^t dt' f(t') P_{n-1}(t-t') = \frac{(t/\tau)^n}{n!}e^{-t/\tau}$.