Spectral Densities with Bounded Support [nln100]

Consider spectral densities with convergent Δ_k-sequences.

Bandwidth:

If $\lim_{k\to\infty} \Delta_k = \frac{1}{4}\omega_0^2$ then $\Phi_0(\omega)$ has compact support on the interval $|\omega| \leq \omega_0$.

Band edge singularity:

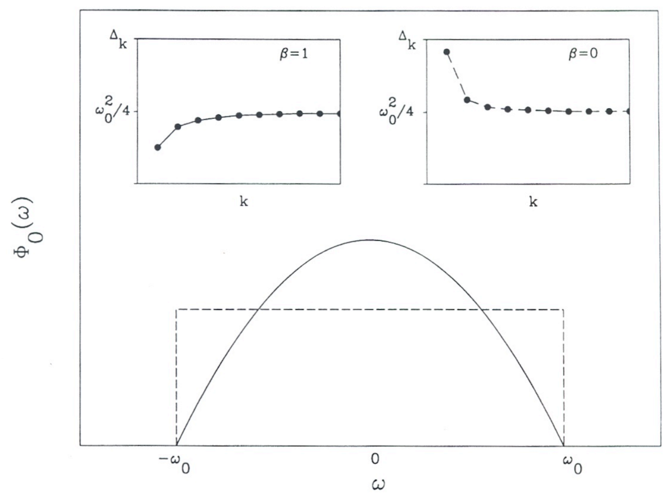
Model spectral density with singularities only at the band edges:[1]

$$\Phi_0(\omega) = \frac{2\pi\omega_0^{2\beta+1}}{B(1/2, 1+\beta)} \left(\omega_0^2 - \omega^2\right)^\beta, \quad |\omega| < \omega_0, \quad \beta > -1. \tag{1}$$

Associated Δ_k-sequence [Magnus 1985] and its asymptotic expansion:

$$\Delta_k = \frac{\omega_0^2 k(k+2\beta)}{(2k+2\beta-1)(2k+2\beta+1)} = \frac{1}{4}\omega_0^2 \left[1 + \frac{1-4\beta^2}{4k^2} + \cdots\right]. \tag{2}$$

Graphical representations for two cases [Viswanath and Müller 1994]:



Analysis of limiting cases in [nex69]:

- $\beta = \frac{1}{2}$: $\Delta_1 = \Delta_2 = \ldots = \frac{1}{4}\omega_0^2 \quad \Rightarrow \Phi_0(\omega) = \frac{4}{\omega_0^2}\sqrt{\omega_0^2 - \omega^2}$.
- $\beta = -\frac{1}{2}$: $\Delta_1 = \frac{1}{2}\omega_0^2,\ \Delta_2 = \Delta_3 \ldots = \frac{1}{4}\omega_0^2 \quad \Rightarrow \Phi_0(\omega) = \frac{2}{\sqrt{\omega_0^2 - \omega^2}}$.

[1] $B(x,y) \doteq \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

Infrared singularity:

Model spectral density with infrared singularity added:

$$\Phi_0(\omega) = \frac{2\pi\omega_0^{-(\alpha+2\beta+1)}}{B\big((1+\alpha)/2, 1+\beta\big)}|\omega|^\alpha(\omega_0^2-\omega^2)^\beta, \quad |\omega|<\omega_0, \quad \alpha,\beta>-1. \quad (3)$$

Associated Δ_k-sequence [Magnus 1985]:

$$\Delta_{2k} = \frac{4\omega_0^2 k(k+\beta)}{(4k+2\beta+\alpha-1)(4k+2\beta+\alpha+1)},$$
$$\Delta_{2k+1} = \frac{\omega_0^2(2k+\alpha+1)(2k+2\beta+\alpha+1)}{(4k+2\beta+\alpha+1)(4k+2\beta+\alpha+3)}. \quad (4)$$

Asymptotic expansion:

$$\sqrt{\Delta_k} = \frac{1}{2}\omega_0\left[1-(-1)^k\frac{\alpha}{2k}+\frac{1-4\beta^2+2(-1)^k\alpha(2\beta+\alpha)}{8k^2}+\cdots\right]. \quad (5)$$

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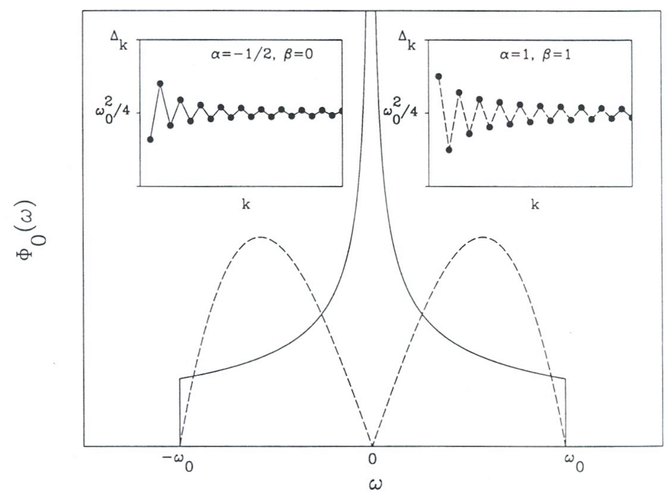

Signature of divergent infrared singularity ($\alpha < 0$): the Δ_{2k+1} converge from below and the Δ_{2k} from above toward the same limit.