Master Equation with detailed balance [nln12]

Master equation with time-independent transition rates $W(n|m) = W_{mn}$:

$$\frac{\partial}{\partial t}P(n,t) = \sum_m [W_{mn}P(m,t) - W_{nm}P(n,t)] = \sum_m L_{mn}P(m,t),$$

where $L_{mn} = W_{mn} - \delta_{mn}\sum_{n'} W_{nn'} = W_{mn} - \delta_{mn}$.

This set of linear, ordinary differential equations can be transformed into an eigenvalue problem with the ansatz $P(m,t) = \varphi_m e^{-\lambda t}$:

Left eigenvector problem: $\sum_m L_{mn}\varphi_m^{(\alpha)} = -\lambda^{(\alpha)}\varphi_n^{(\alpha)}, \quad \alpha = 1, 2, \ldots$

Right eigenvector problem: $\sum_n L_{mn}\chi_n^{(\alpha)} = -\lambda^{(\alpha)}\chi_m^{(\alpha)}, \quad \alpha = 1, 2, \ldots$

Biorthonormality: $\vec{\varphi}^{(\alpha)} \cdot \vec{\chi}^{(\beta)} = \sum_n \varphi_n^{(\alpha)}\chi_n^{(\beta)} = \delta_{\alpha\beta}$.

A stationary solution $P(n)$ requires the existence of a solution of the eigenvalue problem with $\lambda = 0$. The stability of $P(n)$ requires that all other eigenvalues λ have positive real parts.

Detailed balance condition: $W_{mn}P(m) = W_{nm}P(n)$.

Symmetric matrix: $S_{mn} \doteq L_{mn}\sqrt{\frac{P(m)}{P(n)}}$.

Symmetrized eigenvalue problem:

$$\bar{\varphi}_n^{(\alpha)} \doteq \frac{1}{\sqrt{P(n)}}\,\varphi_n^{(\alpha)} \quad \Rightarrow \quad \sum_m S_{mn}\bar{\varphi}_m^{(\alpha)} = -\lambda^{(\alpha)}\bar{\varphi}_n^{(\alpha)}.$$

$$\bar{\chi}_n^{(\alpha)} \doteq \sqrt{P(n)}\,\chi_n^{(\alpha)} \quad \Rightarrow \quad \sum_n S_{mn}\bar{\chi}_n^{(\alpha)} = -\lambda^{(\alpha)}\bar{\chi}_m^{(\alpha)}.$$

Given that $\bar{\varphi}_n^{(\alpha)} = \bar{\chi}_n^{(\alpha)}$ it follows that $\varphi_n^{(\alpha)} = P(n)\chi_n^{(\alpha)}$.

Given that $\sum_n L_{mn} = 0$ it follows that the right eigenvector of L_{mn} with eigenvalue $\lambda = 0$ has components $\chi_n = 1$. The corresponding left eigenvector then has components $\varphi_n = P(n)$.

The symmetric matrix $\mathbf{S}$ has only real, non-negative eigenvalues. Hence $\lambda = 0$ is the smallest eigenvalue. Variational methods are applicable.