Ergodicity [nln13]

Consider a stationary process $x(t)$.
Quantities of interest are expectation values related to $x(t)$.

- Theoretically, we determine *ensemble* averages:
  $\langle x(t)\rangle$, $\langle x^2(t)\rangle$, $\langle x(t)x(t+\tau)\rangle$ are independent of t.
- Experimentally, we determine *time* averages:
  $\overline{x(t)}$, $\overline{x^2(t)}$, $\overline{x(t)x(t+\tau)}$ are independent of t.

Ergodicity: time averages are equal to ensemble averages.

Implication: the ensemble average of a time average has zero variance.

The consequences for the correlation function

$$C(t_1 - t_2) \doteq \langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle\langle x(t_2)\rangle$$

are as follows (set $\tau = t_2 - t_1$ and $t = t_1$):

$$\begin{aligned}
\langle\overline{x^2}\rangle - \langle\overline{x}\rangle^2 &= \lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{+T} dt_1 \int_{-T}^{+T} dt_2\,[\langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle\langle x(t_2)\rangle] \\
&= \lim_{T\to\infty}\frac{1}{4T^2}\int_{-2T}^{+2T} d\tau\, C(\tau)(2T - |\tau|) \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{+2T} d\tau\, C(\tau)\left(1 - \frac{|\tau|}{2T}\right) = 0.
\end{aligned}$$

Necessary condition: $\lim_{\tau\to\infty} C(\tau) = 0$.

Sufficient condition: $\int_0^\infty d\tau\, C(\tau) < \infty$.