Intensity spectrum and spectral density [nln14]

Consider an ergodic process $x(t)$ with $\langle x \rangle = 0$.

Fourier amplitude: $\tilde{x}(\omega, T) \doteq \int_0^T dt\, e^{i\omega t} x(t) \quad \Rightarrow \; \tilde{x}(-\omega, T) = \tilde{x}^*(\omega, T)$.

Intensity spectrum (power spectrum): $I_{xx}(\omega) \doteq \lim_{T\to\infty} \frac{1}{T} \left|\tilde{x}(\omega, T)\right|^2$.

Correlation function: $C_{xx}(\tau) \doteq \langle x(t)x(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, x(t)x(t+\tau)$.

Spectral density: $S_{xx}(\omega) \doteq \int_{-\infty}^{+\infty} d\tau\, e^{i\omega\tau} C_{xx}(\tau)$.

Wiener-Khintchine theorem: $I_{xx}(\omega) = S_{xx}(\omega)$.

Proof:

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\begin{aligned}
I_{xx}(\omega) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T dt' e^{-i\omega t'} x(t') \int_0^T dt\, e^{i\omega t} x(t) \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T d\tau \left[e^{i\omega\tau} \int_0^{T-\tau} dt' x(t')x(t'+\tau) + e^{-i\omega\tau} \int_0^{T-\tau} dt\, x(t)x(t+\tau) \right] \\
&= \lim_{T\to\infty} 2 \int_0^T d\tau\, \cos\omega\tau\, \frac{1}{T} \int_0^{T-\tau} dt\, x(t)x(t+\tau) \\
&= 2 \int_0^\infty d\tau\, \cos\omega\tau C_{xx}(\tau) = \int_{-\infty}^{+\infty} d\tau\, e^{i\omega\tau} C_{xx}(\tau) = S_{xx}(\omega).
\end{aligned}
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