Birth-death master equation: stationary state [nln17]

Master equation: $\dfrac{d}{dt}P(n,t) = \sum_m \Big[W(n|m)P(m,t) - W(m|n)P(n,t)\Big].$

Transition rates: $W(m|n) = \underbrace{T_+(n)\delta_{m,n+1}}_{\text{birth rate}} + \underbrace{T_-(n)\delta_{m,n-1}}_{\text{death rate}}.$

$$\Rightarrow \frac{d}{dt}P(n,t) = T_+(n-1)P(n-1,t) + T_-(n+1)P(n+1,t) - \big[T_+(n) + T_-(n)\big]P(n,t).$$

Stationary state: $P(n,\infty) = P_s(n)$.

Detailed-balance condition: $T_-(n)P_s(n) = T_+(n-1)P_s(n-1), \quad n = 0, 1, 2, \ldots$

Recurrence relation: $P_s(n) = \dfrac{T_+(n-1)}{T_-(n)}\, P_s(n-1).$

Prerequisites:

- $T_-(0) = 0$ (no further deaths at zero population),
- $T_+(0) > 0$ (spontaneous birth from nothing must be permitted if death of last individual is permitted).

Solution: $P_s(n) = P_s(0) \displaystyle\prod_{m=1}^{n} \frac{T_+(m-1)}{T_-(m)}.$

Probability of zero population, $P_s(0)$, determined by normalization condition:

$$\sum_{n=0}^{\infty} P_s(n) = 1.$$

Condition for extreme values (e.g. peak position) in $P_s(n)$:

$$T_+(n-1) = T_-(n).$$