Birth and death of single species [nln19]

Class of processes described by a master equation for some discrete variable n with nonzero transition rates $W(m|n)$ limited to $m = n+1$ and $m = n-1$:

$$\frac{d}{dt}P(n,t) = \sum_m \Big[W(n|m)P(m,t) - W(m|n)P(n,t)\Big],$$

$$W(m|n) = \underbrace{T_+(n)\delta_{m,n+1}}_{\text{birth rate}} + \underbrace{T_-(n)\delta_{m,n-1}}_{\text{death rate}}.$$

The master equation is a difference-differential equation. If $T_\pm(n)$ are polynomials, the master equation can be converted into a linear PDE for the generating function $G(z,t) \doteq \sum_n z^n P(n,t)$:

$$\frac{\partial}{\partial t}G(z,t) = \sum_{l=0}^{L} A_l(z)\frac{\partial^l}{\partial z^l}G(z,t),$$

where L is the highest polynomial order in $T_\pm(n)$.

The notion of nonlinear birth/death rates pertains to quadratic or higher-order terms in $T_\pm(n)$. The PDE for $G(z,t)$ and the master equation for $P(n,t)$ remain linear. The relative ease of solving *linear* birth-death processes is associated with the relative ease of solving *first-order* linear PDEs.

In the context of a deterministic description of the time evolution, nonlinear birth/death rates translate into nonlinear differential equations.

Not all choices of transition rates $T_\pm(n)$ permit a stationary solution,

$$\lim_{t\to\infty} P(n,t) = P_s(n).$$

- Runaway populations can be held in check by death rates that are of higher polynomial order than the birth rates.
- Extinction of populations can be held in check by allowing births out of zero population.