Brownian motion and Gaussian white noise

**Gaussian white noise:** completely factorizing stationary process.

- $P_w(y_1, t_1; y_2, t_2) = P_w(y_1)P_w(y_2)$ if $t_2 \neq t_1$ (factorizability)
- $P_w(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$ (Gaussian nature)
- $\langle y(t) \rangle = 0$ (no bias)
- $\langle y(t_1)y(t_2) \rangle = I_w \delta(t_1 - t_2)$ (whiteness)
- $I_w = \langle y^2 \rangle = \sigma^2$ (intensity)

**Brownian motion:** Markov process.

- Discrete time scale: $t_n = n \Delta t$.
- Position of Brownian particle at time $t_n$: $z_n$.
- $P(z_n, t_n) = \underbrace{P(z_n, t_n|z_{n-1}, t_{n-1})P(z_{n-1}, t_{n-1})}_{P_w(y_n)\delta(y_n-[z_n-z_{n-1}])}$ (white-noise transition rate).
- Specification of white-noise intensity: $I_w = 2D\Delta t$.
- Sample path of Brownian particle: $z(t_n) = \sum_{i=1}^{n} y(t_i)$.
- Position of Brownian particle:
  - mean value: $\langle z(t_n) \rangle = \sum_{i=1}^{n} \langle y(t_i) \rangle = 0$.
  - variance: $\langle z^2(t_n) \rangle = \sum_{i,j=1}^{n} \langle y(t_i)y(t_j) \rangle = 2Dn\Delta t = 2Dt_n$.

Gaussian white noise with intensity $I_w = 2D\Delta t$ is used here to generate the diffusion process discussed previously [nex26], [nex27], [nex97]:

$$P(z, t + \Delta t|z_0, t) = \frac{1}{\sqrt{4\pi D\Delta t}} \exp\left(-\frac{(z - z_0)^2}{4D\Delta t}\right).$$

Sample paths of the diffusion process become continuous in the limit $\Delta t \to 0$ (Lindeberg condition). However, in the present context, we must use $\Delta t \gg \tau_R$, where $\tau_R$ is the relaxation time for the velocity of the Brownian particle.

On this level of contraction, the velocity of the Brownian particle is nowhere defined in agreement with the result of [nex99] that the diffusion process is nowhere differentiable.