Brownian motion and Gaussian white noise [nln20]

Gaussian white noise: completely factorizing stationary process.

- $P_w(y_1, t_1; y_2, t_2) = P_w(y_1)P_w(y_2)$ if $t_2 \neq t_1$ (factorizability)
- $P_w(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$ (Gaussian nature)
- $\langle y(t) \rangle = 0$ (no bias)
- $\langle y(t_1)y(t_2) \rangle = I_w \delta(t_1 - t_2)$ (whiteness)
- $I_w = \langle y^2 \rangle = \sigma^2$ (intensity)

Brownian motion: Markov process.

- Discrete time scale: $t_n = n\,dt$.
- Position of Brownian particle at time t_n: z_n.
- $P(z_n, t_n) = \underbrace{P(z_n, t_n|z_{n-1}, t_{n-1})}_{P_w(y_n)\delta(y_n-[z_n-z_{n-1}])} P(z_{n-1}, t_{n-1})$ (white-noise transition rate).
- Specification of white-noise intensity: $I_w = 2Ddt$.
- Sample path of Brownian particle: $z(t_n) = \sum_{i=1}^{n} y(t_i)$.
- Position of Brownian particle:
  - mean value: $\langle z(t_n) \rangle = \sum_{i=1}^{n} \langle y(t_i) \rangle = 0$.
  - variance: $\langle z^2(t_n) \rangle = \sum_{i,j=1}^{n} \langle y(t_i)y(t_j) \rangle = 2Dndt = 2Dt_n$.

Gaussian white noise with intensity $I_w = 2Ddt$ is used here to generate the diffusion process discussed previously [nex26], [nex27], [nex97]:

$$P(z, t + dt|z_0, t) = \frac{1}{\sqrt{4\pi Ddt}} \exp\left(-\frac{(z - z_0)^2}{4Ddt}\right).$$

Sample paths of the diffusion process become continuous in the limit $dt \to 0$ (Lindeberg condition). However, in the present context, we must use $dt \gg \tau_R$, where τ_R is the relaxation time for the velocity of the Brownian particle.

On this level of contraction, the velocity of the Brownian particle is nowhere defined in agreement with the result of [nex99] that the diffusion process is nowhere differentiable.