Kubo formula for response function [nln27]

Interaction representation for time evolution of $\mathcal{H}(t) = \mathcal{H}_0 - b(t)B$:

$$\frac{dA}{dt} = \frac{\imath}{\hbar}[\mathcal{H}_0, A] \quad \Rightarrow \; A(t) = e^{\imath\mathcal{H}_0 t/\hbar} A e^{-\imath\mathcal{H}_0 t/\hbar},$$

$$\frac{dB}{dt} = \frac{\imath}{\hbar}[\mathcal{H}_0, B] \quad \Rightarrow \; B(t) = e^{\imath\mathcal{H}_0 t/\hbar} B e^{-\imath\mathcal{H}_0 t/\hbar},$$

$$\frac{d\rho}{dt} = -\frac{\imath}{\hbar}[-b(t)B, \rho] \quad \Rightarrow \; \rho(t) = \rho_0 + \frac{\imath}{\hbar}\int_{-\infty}^{t} dt' b(t')[B(t'), \rho(t')].$$

Set $\rho(t) = \rho_0 + \rho_1(t)$ with $\rho_0 = Z_0^{-1} e^{-\beta\mathcal{H}_0}$.

Full response: $\langle A(t)\rangle - \langle A\rangle_0 = \mathrm{Tr}\{\rho_1(t)A(t)\}$

Leading correction to ρ_0: $\rho_1(t) \simeq \frac{\imath}{\hbar}\int_{-\infty}^{t} dt' b(t')[B(t'), \rho_0]$

Linear response:

$$\begin{aligned}\langle A(t)\rangle - \langle A\rangle_0 &= \frac{\imath}{\hbar}\int_{-\infty}^{t} dt' b(t')\mathrm{Tr}\{[B(t'), \rho_0]A(t)\} \\ &= \frac{\imath}{\hbar}\int_{-\infty}^{t} dt' b(t')\mathrm{Tr}\{\rho_0[A(t), B(t')]\} \\ &= \frac{\imath}{\hbar}\int_{-\infty}^{t} dt' b(t')\langle[A(t), B(t')]\rangle_0.\end{aligned}$$

Compare with definition of response function in [nln26].

Kubo formula:

$$\tilde{\chi}_{AB}(t-t') = \frac{\imath}{\hbar}\theta(t-t')\langle[A(t), B(t')]\rangle_0.$$

- Causality requirement is ensured by step function $\theta(t-t')$.
- Hermitian A, B imply Hermitian $\imath[A, B]$. Hence $\tilde{\chi}(t)$ is real.
- Linear response depends only on equilibrium quantities.
- Response function only depends on time difference $t - t'$.

The Kubo formula establishes a general link between

- the dynamical properties of a many-body system at equilibrium,
- the dynamical response of that system to experimental probes.