First projection [nln34]

Rewrite relaxation function from [nln31] with projection operators from [nln33] and apply Dyson identity $(X+Y)^{-1} = X^{-1} - X^{-1}Y(X+Y)^{-1}$:[1]

$$\begin{aligned}\langle f_0|f_0\rangle c_0(z) &= \left\langle f_0 \left| \frac{1}{z+\imath L} \right| f_0 \right\rangle = \left\langle f_0 \left| \frac{1}{z+\imath LP_0+\imath LQ_0} \right| f_0 \right\rangle \\ &= \left\langle f_0 \left| \frac{1}{z+\imath LQ_0} \right| f_0 \right\rangle + \left\langle f_0 \left| \frac{1}{z+\imath LQ_0}\, \imath LP_0\, \frac{1}{z+\imath L} \right| f_0 \right\rangle.\end{aligned}$$

Simplify both terms:

- $\left\langle f_0 \left| \frac{1}{z}\left[1 + \frac{(-\imath)}{z}LQ_0 + \frac{(-\imath)^2}{z^2}LQ_0LQ_0 + \cdots\right] \right| f_0 \right\rangle = \frac{1}{z}\langle f_0|f_0\rangle.$
- $\left\langle f_0 \left| \frac{1}{z+\imath LQ_0}\imath L \right| f_0 \right\rangle \frac{1}{\langle f_0|f_0\rangle} \left\langle f_0 \left| \frac{1}{z+\imath L} \right| f_0 \right\rangle = \left\langle f_0 \left| \frac{1}{z+\imath LQ_0}\imath L \right| f_0 \right\rangle c_0(z).$

$$\Rightarrow\ c_0(z) = \frac{1}{z + \frac{1}{\langle f_0|f_0\rangle}\left\langle f_0 \left| \frac{z}{z+\imath LQ_0}\imath L \right| f_0 \right\rangle},$$

$$\begin{aligned}\left\langle f_0 \left| \frac{z}{z+\imath LQ_0}\imath L \right| f_0 \right\rangle &= \left\langle f_0 \left| \left[1 - (z+\imath LQ_0)\frac{1}{z+\imath LQ_0} + \frac{z}{z+\imath LQ_0}\right]\imath L \right| f_0 \right\rangle \\ &= \left\langle f_0 \left| (-\imath)LQ_0\frac{1}{z+\imath LQ_0}\imath L \right| f_0 \right\rangle = \left\langle f_0 \left| (-\imath)LQ_0\frac{1}{z+\imath Q_0LQ_0}Q_0\imath L \right| f_0 \right\rangle \\ &= \left\langle f_1 \left| \frac{1}{z+\imath L_1} \right| f_1 \right\rangle; \qquad |f_1\rangle = Q_0|f_1\rangle = Q_0\imath L|f_0\rangle, \quad L_1 = Q_0LQ_0.\end{aligned}$$

Relaxation function after first projection expressed via memory function:

$$c_0(z) = \frac{1}{z+\Sigma_1(z)}, \qquad \Sigma_1(z) = \frac{1}{\langle f_0|f_0\rangle}\left\langle f_1 \left| \frac{1}{z+\imath L_1} \right| f_1 \right\rangle.$$

Memory function $\Sigma_1(z)$ of original problem, $\{L, |f_0\rangle\}$, can be reinterpreted as the (as yet non-normalized) relaxation function of a new dynamical problem, $\{L_1, |f_1\rangle\}$.

Projection operator Q_0 acts as filter on the Liouvillian L, absorbing that part of dynamics dealt with explicitly in first projection. Explicit information contained in normalization constant of $\Sigma_1(z)$.

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[1] Direct consequence of operator identity $(X+Y)(X+Y)^{-1} = X(X+Y)^{-1} + Y(X+Y)^{-1} = 1$, here with $X \doteq z + \imath LQ_0, \quad Y \doteq \imath LP_0$.