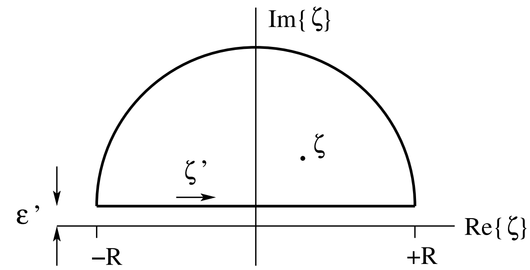
Kramers-Kronig dispersion relations [nln37]

Use analyticity of $\chi_{AA}(\zeta)$ for $\Im\{\zeta\} > 0$.

Cauchy integral: $\chi_{AA}(\zeta) = \dfrac{1}{2\pi\imath}\displaystyle\int_{\mathcal{C}} d\zeta' \frac{\chi_{AA}(\zeta')}{\zeta' - \zeta}$.



Integral converges for $\zeta' = \omega' + \imath\epsilon'$, $\epsilon' \to 0$.
Integral along semi-circle vanishes for $R \to \infty$:
Sum rule implies $\chi_{AA}(\zeta) \lesssim |\zeta|^{-1}$ for $|\zeta| \to \infty$.

$$\Rightarrow\ \chi_{AA}(\zeta) = \frac{1}{2\pi\imath}\int_{-\infty}^{+\infty} d\omega' \frac{\chi_{AA}(\omega')}{\omega' - \zeta}.$$

Set $\zeta = \omega + \imath\epsilon$ and use $\displaystyle\lim_{\epsilon\to 0}\frac{1}{\omega' - \omega \mp \imath\epsilon} = \mathrm{P}\frac{1}{\omega' - \omega} \pm \imath\pi\delta(\omega' - \omega)$.

$$\begin{aligned}\chi_{AA}(\omega) &= \lim_{\epsilon\to 0}\chi_{AA}(\omega + \imath\epsilon) = \lim_{\epsilon\to 0}\frac{1}{2\pi\imath}\int_{-\infty}^{+\infty} d\omega' \frac{\chi_{AA}(\omega')}{\omega' - \omega - \imath\epsilon} \\ &= \frac{1}{2\pi\imath}\,\mathrm{P}\!\int_{-\infty}^{+\infty} d\omega' \frac{\chi_{AA}(\omega')}{\omega' - \omega} + \underbrace{\frac{1}{2}\int_{-\infty}^{+\infty} d\omega' \chi_{AA}(\omega')\delta(\omega' - \omega)}_{\frac{1}{2}\chi_{AA}(\omega)}.\end{aligned}$$

$$\Rightarrow\ \chi_{AA}(\omega) \doteq \chi'_{AA}(\omega) + \imath\chi''_{AA}(\omega) = \frac{1}{\pi\imath}\,\mathrm{P}\!\int_{-\infty}^{+\infty} d\omega' \frac{\chi_{AA}(\omega')}{\omega' - \omega}.$$

Consider real and imaginary parts of this relation separately:

$$\chi'_{AA}(\omega) = \frac{1}{\pi}\,\mathrm{P}\!\int_{-\infty}^{+\infty} d\omega' \frac{\chi''_{AA}(\omega')}{\omega' - \omega}, \qquad \chi''_{AA}(\omega) = -\frac{1}{\pi}\,\mathrm{P}\!\int_{-\infty}^{+\infty} d\omega' \frac{\chi'_{AA}(\omega')}{\omega' - \omega}.$$

The Kramers-Kronig relations are a consequence of the causality property of the response function.