Generating function

The generating function $G_X(z)$ is a representation of the characteristic function $\Phi_X(k)$ that is most commonly used, along with factorial moments and factorial cumulants, if the stochastic variable $X$ is integer valued.

Definition: $G_X(z) \doteq \langle z^x \rangle$ with $|z| = 1$.

Application to continuous and discrete (integer-valued) stochastic variables:

$$G_X(z) = \int dx \, z^x P_X(x), \quad G_X(z) = \sum_n z^n P_X(n).$$

Definition of factorial moments:

$$\langle X^m \rangle_f \doteq \langle X(X - 1) \cdots (X - m + 1) \rangle, \quad m \geq 1; \quad \langle X^0 \rangle_f \doteq 0.$$

Function generating factorial moments:

$$G_X(z) = \sum_{m=0}^{\infty} \frac{(z - 1)^m}{m!} \langle X^m \rangle_f, \quad \langle X^m \rangle_f = \left. \frac{d^m}{dz^m} G_X(z) \right|_{z=1}.$$

Function generating factorial cumulants:

$$\ln G_X(z) = \sum_{m=1}^{\infty} \frac{(z - 1)^m}{m!} \langle \langle X^m \rangle \rangle_f, \quad \langle \langle X^m \rangle \rangle_f = \left. \frac{d^m}{dz^m} \ln G_X(z) \right|_{z=1}.$$

Applications:

▷ Moments and cumulants of the Poisson distribution [nex16]
▷ Pascal distribution [nex22]
▷ Reconstructing probability distributions [nex14]