Time-Dependent Probability Distributions

Stochastic processes describe systems evolving probabilistically in time. Consider a process characterized by the stochastic variable $X(t)$. In general, the time evolution of $X(t)$ is encoded in a hierarchy of time-dependent joint probability distributions:

$$P(x_1, t_1), \ P(x_1, t_1; x_2, t_2), \ldots, \ P(x_1, t_1; x_2, t_2; \ldots; x_n t_n), \ldots$$

with time-ordering $t_1 \geq t_2 \geq \ldots \geq t_n \geq \ldots$ implied.

Attributes:

- $P(x_1, t_1; x_2, t_2; \ldots) \geq 0,$
- $\int dx_1 P(x_1, t_1; x_2, t_2; \ldots) = P(x_2, t_2; \ldots),$
- $\int dx_1 P(x_1, t_1) = 1.$

Conditional probability distribution: $P(x_1, t_1|x_2, t_2) = \frac{P(x_1, t_1; x_2, t_2)}{P(x_2, t_2)}.$

Attributes: $\int dx_2 P(x_1, t_1|x_2, t_2) = P(x_1, t_1), \ \int dx_1 P(x_1, t_1|x_2, t_2) = 1.$

More generally: $P(x_1, t_1; \ldots|x_1, t_1; \ldots) = \frac{P(x_1, t_1; \ldots; x_1, t_1; \ldots)}{P(x_1, t_1; \ldots)}.$

Autocorrelation function: $\langle X(t_1)X(t_2) \rangle = \int dx_1 \int dx_2 x_1 x_2 P(x_1, t_1; x_2, t_2).$

More generally:

$$\langle [X(t_1)]^{m_1} \cdots [X(t_n)]^{m_n} \rangle = \int dx_1 \cdots \int dx_n x_1^{m_1} \cdots x_n^{m_n} P(x_1, t_1; \ldots; x_n, t_n).$$

Stationary processes:

- $P(x_1, t_1; x_2, t_2; \ldots) = P(x_1, t_1 + \tau; x_2, t_2 + \tau; \ldots)$ for any $\tau,$
- $P(x_1, t_1) = P(x_1, 0)$ (time-independent),
- $P(x_1, t_1|x_2, t_2) = P(x_1, t_1 - t_2|x_2, 0)$ (initial condition),
- $\langle X(t_1)X(t_2) \rangle = \langle X(t_1 - t_2)X(0) \rangle.$

Equilibrium implies stationarity but not vice versa.