Time-Dependent Probability Distributions [nln50]

Stochastic processes describe systems evolving probabilistically in time. Consider a process characterized by the stochastic variable $X(t)$. In general, the time evolution of $X(t)$ is encoded in a hierarchy of time-dependent joint probability distributions:

$$P(x_1, t_1), \quad P(x_1, t_1; x_2, t_2), \quad \ldots, \quad P(x_1, t_1; x_2, t_2; \ldots; x_n t_n), \quad \ldots$$

with time-ordering $t_1 \geq t_2 \geq \cdots \geq t_n \geq \cdots$ implied.

Attributes:

- $P(x_1, t_1; x_2, t_2; \ldots) \geq 0$,
- $\displaystyle\int dx_1 P(x_1, t_1; x_2, t_2, \ldots) = P(x_2, t_2; \ldots)$,
- $\displaystyle\int dx_1 P(x_1, t_1) = 1$.

Conditional probability distribution: $\displaystyle P(x_1, t_1|x_2, t_2) = \frac{P(x_1, t_1; x_2, t_2)}{P(x_2, t_2)}$.

Attributes: $\displaystyle\int dx_2 P(x_1, t_1|x_2, t_2) = P(x_1, t_1), \quad \int dx_1 P(x_1, t_1|x_2, t_2) = 1$.

More generally: $\displaystyle P(x_1, t_1; \ldots|\bar{x}_1, \bar{t}_1; \ldots) = \frac{P(x_1, t_1; \ldots; \bar{x}_1, \bar{t}_1; \ldots)}{P(\bar{x}_1, \bar{t}_1; \ldots)}$.

Autocorrelation function: $\displaystyle\langle X(t_1)X(t_2)\rangle = \int dx_1 \int dx_2\, x_1 x_2 P(x_1, t_1; x_2, t_2)$.

More generally:

$$\Big\langle [X(t_1)]^{m_1} \cdots [X(t_n)]^{m_n} \Big\rangle = \int dx_1 \cdots \int dx_n\, x_1^{m_1} \cdots x_n^{m_n} P(x_1, t_1; \ldots; x_n, t_n).$$

Stationary processes:

- $P(x_1, t_1; x_2, t_2; \ldots) = P(x_1, t_1 + \tau; x_2, t_2 + \tau; \ldots)$ for any τ,
- $P(x_1, t_1) = P(x_1, 0)$ (time-independent),
- $P(x_1, t_1|x_2, t_2) = P(x_1, t_1 - t_2|x_2, 0)$ (initial condition),
- $\langle X(t_1)X(t_2)\rangle = \langle X(t_1 - t_2)X(0)\rangle$.

Equilibrium implies stationarity but not vice versa.