Markov Process: General Attributes [nln54]

Specification of Markov process:

- $P(x, t_0)$ (initial probability distribution),
- $P(x_1, t_1|x_2, t_2)$ (conditional probability distribution).

The entire hierarchy of joint probability distributions (see [nln50]) can be generated from these two ingredients if the process is Markovian.

Two times $t_1 \geq t_2$:

$$P(x_1, t_1; x_2, t_2) = P(x_1, t_1|x_2, t_2)P(x_2, t_2).$$

Three times $t_1 \geq t_2 \geq t_3$:

$$\begin{aligned} P(x_1, t_1; x_2, t_2; x_3, t_3) &= P(x_1, t_1; x_2, t_2|x_3, t_3)P(x_3, t_3) \\ &= P(x_1, t_1|x_2, t_2; x_3, t_3)P(x_2, t_2|x_3, t_3)P(x_3, t_3) \\ &= P(x_1, t_1|x_2, t_2)P(x_2, t_2|x_3, t_3)P(x_3, t_3). \end{aligned}$$

Comments:

- The step from the first to the second line uses the previous equation in a reduced sample space (specified by one condition).
- The second condition in the middle line is redundant.

Integration over the variable x_2 at intermediate time t_2 yields

$$\underbrace{P(x_1, t_1; x_3, t_3)}_{P(x_1, t_1|x_3, t_3)P(x_3, t_3)} = P(x_3, t_3)\int dx_2 P(x_1, t_1|x_2, t_2)P(x_2, t_2|x_3, t_3).$$

Division by $P(x_3, t_3)$ then yields the *Chapman-Kolmogorov* equation:

$$P(x_1, t_1|x_3, t_3) = \int dx_2 P(x_1, t_1|x_2, t_2)P(x_2, t_2|x_3, t_3), \quad t_1 \geq t_2 \geq t_3.$$

The Chapman-Kolmogorov equation is a functional equation between conditional probability distributions with many different kinds of solutions.

Put differently ...

Any two non-negative and normalized functions $P(x,t)$ and $P(x_1,t_1|x_2,t_2)$ represent a unique Markov process if they satisfy the following two conditions:

- $P(x_1,t_1) = \int dx_2 P(x_1,t_1|x_2,t_2)P(x_2,t_2) \quad (t_1 \geq t_2)$,
- $P(x_1,t_1|x_3,t_3) = \int dx_2 P(x_1,t_1|x_2,t_2)P(x_2,t_2|x_3,t_3) \quad (t_1 \geq t_2 \geq t_3)$.

The first condition implies that $\lim_{\Delta t \to 0} P(x_1, t+\Delta t|x_2,t) = \delta(x_1 - x_2)$.

Homogeneous process: $P(x_1, t+\Delta t|x_0,t) \doteq P(x_1|x_0;\Delta t)$ independent of t.

The two conditions thus become

- $P(x_1, t+\Delta t) = \int dx_2 P(x_1|x_2;\Delta t)P(x_2,t)$,
- $P(x_1|x_3;\Delta t_{13}) = \int dx_2 P(x_1|x_2;\Delta t_{12})P(x_2|x_3;\Delta t_{23})$
  with $\Delta t_{13} = \Delta t_{12} + \Delta t_{23}$.

For initial condition $P(x,0) = \delta(x - x_0)$ we then have $P(x,t) = P(x|x_0;t)$.