Diffusion Process and Cauchy Process [nln55]

Here we portray two of the most common homogeneous Markov processes.

- ☐ Diffusion process: $P(x|x_0;\Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}} \exp\left(-\frac{(x-x_0)^2}{4D\Delta t}\right)$.
- ☐ Cauchy process: $P(x|x_0;\Delta t) = \frac{1}{\pi}\frac{\Delta t}{(x-x_0)^2+(\Delta t)^2}$.

Both processes satisfy

- $\int_{-\infty}^{+\infty} dx P(x|x_0;\Delta t) = 1$ (normalization),
- $\lim_{\Delta t\to 0} P(x|x_0;\Delta t) = \delta(x-x_0)$ (consistency),
- $P(x_1|x_3;\Delta t_{13}) = \int dx_2 P(x_1|x_2;\Delta t_{12})P(x_2|x_3;\Delta t_{23})$ (C.-K. eq.) [nex26].

Q: Are the sample paths of the two processes continuous or discontinuous?

A: The sample paths are continuous in the diffusion process and discontinuous in the Cauchy process [nex97].

Lindeberg criterion for continuous sample paths:

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\int_{|x-x_0|>\epsilon} dx\, P(x|x_0;\Delta t) = 0$$

for any $\epsilon > 0$ and uniformly in x_0 and Δt.

Interpretation: the probability for the final position x to be finitely different from the initial position x_0 goes to zero faster than Δt as $\Delta t \to 0$.

Computer generated sample paths for both processes are shown in [nsl1].

Q: Are the sample paths of the two processe differentiable?

A: In both processes the sample paths are nowhere differentiable [nex99].